DRAFT: FINITE ELEMENT SPACES OF DOUBLE FORMS

YAKOV BERCHENKO-KOGAN AND EVAN S. GAWLIK

ABSTRACT. The tensor product of two differential forms of degree p and q is a multilinear form that is alternating in its first p arguments and alternating in its last q arguments. These forms, which are known as double forms or (p, q)-forms, play a central role in certain differential complexes that arise when studying partial differential equations. We construct piecewise constant finite element spaces for all of the natural subspaces of the space of (p, q)-forms, excluding one subspace which fails to admit a piecewise constant discretization. As special cases, our construction recovers known finite element spaces for symmetric matrices with tangential-tangential continuity (the Regge finite elements), symmetric matrices with normal-normal continuity, and trace-free matrices with normal-tangential continuity. It also gives rise to new spaces, like a finite element space for tensors possessing the symmetries of the Riemann curvature tensor.

1. Introduction

Over the last several decades, exterior calculus has played an important role in the development of numerical methods for partial differential equations (PDEs). Notably, Arnold, Falk, and Winther [1, 2] showed that finite element methods for many PDEs can be best understood by viewing the unknowns as differential forms and seeking approximate solutions in finite-dimensional spaces of differential forms. These finite-dimensional spaces, or finite element spaces, consist of differential forms which are piecewise polynomial with respect to a simplicial triangulation of the domain on which the PDE is posed. When chosen carefully, such spaces give rise to stable mixed discretizations of PDEs involving the Hodge-Laplace operator. Arnold, Falk, and Winther's work led to a complete classification of such spaces, generalizing and unifying finite element spaces that are attributed to Whitney [26], Raviart and Thomas [24], Nédélec [22, 23], Brezzi, Douglas, and Marini [6], and others. Their work also highlighted the importance of differential complexes—particularly the de Rham complex—in the design and analysis of finite element methods for PDEs.

In this paper, we construct finite element spaces for *double forms*: tensor products of differential forms. Unlike ordinary differential k-forms, which are multilinear and alternating in all k of their arguments, the tensor product of a p-form and a q-form is a multilinear form that is alternating in its first p arguments and alternating in its last q arguments. These forms, which are known as double forms or (p, q)-forms, have a long history in differential geometry [7, 10, 15–17, 19] and have recently drawn the attention of numerical analysts [3, 5] due to their role in certain differential complexes that arise when studying PDEs.

To be specific, we consider an n-dimensional simplicial triangulation $\mathcal{T}$ and focus on constructing *piecewise constant* (p, q)-forms that are single-valued when restricted to any simplex $\sigma \in \mathcal{T}$ of dimension less than n. Here, restricting a (p, q)-form φ to a simplex σ means that we not only evaluate φ at points on σ, but we also feed into φ only vectors which are tangent to σ. It turns out that the full space of (p, q)-forms does not admit such a discretization (unless $p + q \geq n$); only certain subspaces do. We determine which subspaces admit such a discretization and, for those that do, we construct one by providing degrees of freedom for the finite element space.

Our construction recovers several known finite element spaces as special cases. The piecewise constant Regge finite element space is one example [8, 9, 21]. The members of this space are often described as piecewise constant symmetric matrices possessing tangential-tangential continuity, and the space has one degree of freedom per edge in the triangulation. In our language, the Regge finite elements are symmetric $(1, 1)$-forms with single-valued restrictions to lower-dimensional simplices. The word "symmetric" is important here; our construction recovers the piecewise constant Regge finite element space when considering symmetric $(1, 1)$-forms but fails to provide a finite element space when considering antisymmetric $(1, 1)$-forms (except in dimension $n = 2$). This is consistent with the fact that antisymmetric $(1, 1)$-forms are simply 2-forms, and piecewise constant 2-forms with tangential continuity do not exist in dimension $n > 2$.

In the same way that the space of $(1, 1)$-forms decomposes naturally into two subspaces—symmetric $(1, 1)$-forms and antisymmetric $(1, 1)$-forms—the space of (p, q)-forms admits a natural decomposition as well. This decomposition, which has its origins in representation theory [11, Exercises 6.13* and 15.32*] and involves at most $\min\{p, q\} + 1$ summands, can be characterized in several different ways [12]. For our purposes it is convenient to regard this decomposition as an eigendecomposition of a certain self-adjoint operator on (p, q)-forms. Relative to this decomposition, we show that all but one of the summands admits a piecewise constant discretization. The exceptional summand consists of those (p, q)-forms that alternate in all $p+q$ arguments, i.e. the (p, q)-forms that are actually $(p + q)$-forms.

When one considers $(2, 1)$-forms, there are two summands in the aforementioned decomposition. For one of those summands, our construction yields a finite element space that in 3D is isomorphic to the space of piecewise constant, trace-free matrices with normal-tangential continuity introduced by Gopalakrishnan, Lederer, and Schöberl [14]. This space has two degrees of freedom per triangle.

For $(2, 2)$-forms in dimension $n = 3$, our construction yields (for one of the summands in the decomposition) a finite element space that is isomorphic to the space of piecewise constant, symmetric matrices with normal-normal continuity introduced by Sinwel [25]. This space has one degree of freedom per triangle and two degrees of freedom per tetrahedron.

Our construction also gives rise to many new finite element spaces. Of particular interest are $(2,2)$-forms in dimension $n \geq 3$ that satisfy the algebraic Bianchi identity $\varphi(X,Y;Z,W)+\varphi(Y,Z;X,W)+\varphi(Z,X;Y,W)=0$. These so-called *algebraic curvature tensors* possess precisely the same symmetries as the Riemann curvature tensor from differential geometry (including the symmetry $\varphi(X,Y;Z,W)=\varphi(Z,W;X,Y)$, which follows from the algebraic Bianchi identity and the fact that $\varphi(X,Y;Z,W)$ alternates in X and Y and alternates in Z and W). Our construction yields a piecewise constant finite element space for such algebraic curvature tensors. In dimension $n=3$, the space is the same as the one mentioned above that can be identified with Sinwel's space. In dimension $n>3$, the space has one degree of freedom per triangle and two degrees of freedom per tetrahedron, just like in dimension $n=3$. Let us remark that in certain contexts, it may be preferable to work with a dual version of these double forms, namely $(n-2,n-2)$-forms whose double Hodge dual satisfies the algebraic Bianchi identity. Our construction yields a finite element space for these double forms as well. As discussed in [13], such a finite element space may be useful for computations that involve the distributional Riemann curvature tensor.

1.1. **Organization.** We begin in Section 2 by studying the algebraic structure of multilinear functionals that alternate in their first p arguments and alternate in their last q arguments. We show that these functionals, or (p,q)-covectors, admit a natural decomposition. We then bring spatial dependence into the picture in Section 3 and study (p,q)-forms on a manifold. The tools developed in Sections 2 and 3 will be used to prove a key result in Section 4: Nearly every (p,q)-form on the standard simplex $T^n=\{(\lambda_0,\dots,\lambda_n) \mid \sum_i \lambda_i = 1\}$ with vanishing trace on ∂T^n can be extended to a (p,q)-form on $\mathbb{R}^{n+1}$ with vanishing trace on the coordinate hyperplanes. We show that this extension preserves the aforementioned decomposition, and that such an extension fails to exist for precisely one of the summands in the decomposition. We use this result to prove the existence of piecewise constant finite element spaces for (p,q)-forms in Section 5. These finite element spaces exist for all subspaces in the decomposition except for the one that fails to admit extensions. We conclude Section 5 by providing formulas for the dimensions of the finite element spaces and for the number of degrees of freedom that one must assign to each simplex to ensure unisolvence. We give examples that show how our construction recovers some known finite element spaces and discovers some new ones.

2. Double multicovectors

Definition 2.1. Let V be a finite-dimensional vector space. A *k-covector* or *multicovector* is an antisymmetric k-linear functional on V. The space of

k-covectors is denoted $\bigwedge^k V^*$, which we will abbreviate as Λ^k when there is no risk of confusion.

Definition 2.2. Let $\Lambda^{p,q} := \Lambda^p \otimes \Lambda^q$. When we wish to emphasize the space V, we will use the notation $\bigwedge^{p,q} V^*$. A (p,q)-*covector* or *double multicovector* is an element of $\Lambda^{p,q}$. Letting $k = p + q$, double multicovectors are k-linear functionals on V which are antisymmetric in the first p indices and antisymmetric in the last q indices.

The space of double multicovectors has a much richer structure than the space of regular multicovectors. Ultimately, this rich structure arises because, unlike Λ^k, the space of double covectors $\Lambda^{p,q}$ is *not* an irreducible representation with respect to the natural action of $GL(V)$. Consequently, $\Lambda^{p,q}$ has a natural decomposition into subspaces, and there are nontrivial natural maps between the $\Lambda^{p,q}$. We discuss the connection to representation theory in a forthcoming Appendix. For now, we give a more elementary discussion of this structure.

Notation 2.3. If $e_1, \dots, e_n$ is a basis for V, let $e^1, \dots, e^n$ be the corresponding dual basis of V^*. For a multi-index $I = (i_1, \dots, i_k)$, let $e^I = e^{i_1} \wedge \cdots \wedge e^{i_k} \in \Lambda^k$. Let $e^{I,J} = e^I \otimes e^J \in \Lambda^{p,q}$, where $p = |I|$ and $q = |J|$.

If we restrict I and J to each be in increasing order, then the $e^{I,J}$ form a basis of $\Lambda^{p,q}$. If $e_1, \dots, e_n$ is orthonormal with respect to an inner product on V, then this $e^{I,J}$ basis is orthonormal with respect to the induced inner product on $\Lambda^{p,q}$.

2.1. **The s and s^* operators.**

For any p and q, there is a natural map $s\colon \Lambda^{p,q} \to \Lambda^{p+1,q-1}$. Up to a constant, this map is simply antisymmetrization in the first $p+1$ indices. There is likewise a corresponding map $s^*\colon \Lambda^{p,q} \to \Lambda^{p-1,q+1}$. The decomposition alluded to earlier is simply the eigendecomposition of s^*s. We now discuss these operators and this decomposition in more detail.

Definition 2.4. Let $s\colon \Lambda^{p,q} \to \Lambda^{p+1,q-1}$ denote the map defined by

$$(s\varphi)(X_1, \dots, X_{p+1}; Y_1, \dots, Y_{q-1}) = \sum_{a=1}^{p+1} (-1)^{a-1} \varphi(X_1, \dots, \widehat{X_a}, \dots, X_{p+1}; X_a, Y_1, \dots, Y_{q-1}).$$

Equivalently, we can define s on simple tensors by

$$s\big((\alpha_1 \wedge \cdots \wedge \alpha_p) \otimes (\beta_1 \wedge \cdots \wedge \beta_q)\big) = \sum_{a=1}^{q} (-1)^{a-1} (\beta_a \wedge \alpha_1 \wedge \cdots \wedge \alpha_p) \otimes (\beta_1 \wedge \cdots \wedge \widehat{\beta_a} \wedge \cdots \wedge \beta_q).$$

and then extending by linearity.

See [7, 15, 16] for more discussion of this map. We can likewise define the map going the other way.

Definition 2.5. Let $s^*\colon \Lambda^{p,q} \to \Lambda^{p-1,q+1}$ denote the map defined by

$$(s^*\varphi)(X_1, \dots, X_{p-1}; Y_1, \dots, Y_{q+1}) = \sum_{a=1}^{q+1} (-1)^{a-1} \varphi(Y_a, X_1, \dots, X_{p-1}; Y_1, \dots, \widehat{Y_a}, \dots, Y_{q+1}),$$

or, equivalently, by

$$s^*\big((\alpha_1 \wedge \cdots \wedge \alpha_p) \otimes (\beta_1 \wedge \cdots \wedge \beta_q)\big) = \sum_{a=1}^{p} (-1)^{a-1} (\alpha_1 \wedge \cdots \wedge \widehat{\alpha_a} \wedge \cdots \wedge \alpha_p) \otimes (\alpha_a \wedge \beta_1 \wedge \cdots \wedge \beta_q).$$

As the notation suggests, s and s^* are adjoints of each other, with respect to the natural inner product on double multicovectors induced by an arbitrary inner product on V. We will prove this result shortly.

We can give an alternate characterization of the s operator as wedge-contraction with the identity linear transformation.

Proposition 2.6. *We have*

$$s(\alpha \otimes \beta) = \sum_{i=1}^{n} (e^i \wedge \alpha) \otimes (e_i \lrcorner \beta),$$

$$s^*(\alpha \otimes \beta) = \sum_{i=1}^{n} (e_i \lrcorner \alpha) \otimes (e^i \wedge \beta),$$

where α is a p-covector and β is a q-covector.

Proof. Checking on a basis, we must check that

$$s(e^{I,J}) = \sum_{i=1}^{n} (e^i \wedge e^I) \otimes (e_i \lrcorner e^J).$$

By the above definition, the left-hand side is

$$s(e^{I,J}) = \sum_{a=1}^{q} (-1)^{a-1} (e^{j_a} \wedge e^{i_1} \wedge \cdots \wedge e^{i_p}) \otimes (e^{j_1} \wedge \cdots \wedge \widehat{e^{j_a}} \wedge \cdots \wedge e^{j_q}),$$

where $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$.

Moving on to the right-hand side, we have that $e_i \lrcorner e^J = 0$ unless $i = j_a$ for some a. Thus, we can instead sum over a, obtaining

$$\sum_{i=1}^{n} (e^i \wedge e^I) \otimes (e_i \lrcorner e^J) = \sum_{a=1}^{q} (e^{j_a} \wedge e^I) \otimes (e_{j_a} \lrcorner e^J).$$

The claim follows because $e_{j_a} \lrcorner e^J = (-1)^{a-1}(e^{j_1} \wedge \cdots \wedge \widehat{e^{j_a}} \wedge \cdots \wedge e^{j_q})$. The result for the s^* operator is analogous. $\square$

Assuming that $e_1, \dots, e_n$ is orthonormal, the operators $e_i \lrcorner$ and $e^i \wedge$ are adjoints of each other. We can use this property with the previous proposition to show that the operators s and s^* are adjoints of each other.

Proposition 2.7. *The operators s and s^* are adjoints of each other.*

Proof. It suffices to check on simple tensors that

$$\langle s(\alpha \otimes \beta), \gamma \otimes \delta \rangle = \langle \alpha \otimes \beta, s^*(\gamma \otimes \delta) \rangle ,$$

where $\alpha \in \Lambda^p$, $\beta \in \Lambda^q$, $\gamma \in \Lambda^{p+1}$, and $\delta \in \Lambda^{q-1}$. We compute

$$\begin{aligned} \langle s(\alpha \otimes \beta), \gamma \otimes \delta \rangle &= \sum_{i=1}^n \left\langle (e^i \wedge \alpha) \otimes (e_i \lrcorner \beta), \gamma \otimes \delta \right\rangle \\ &= \sum_{i=1}^n \left\langle e^i \wedge \alpha, \gamma \right\rangle \langle e_i \lrcorner \beta, \delta \rangle \\ &= \sum_{i=1}^n \langle \alpha, e_i \lrcorner \gamma \rangle \left\langle \beta, e^i \wedge \delta \right\rangle \\ &= \sum_{i=1}^n \left\langle \alpha \otimes \beta, (e_i \lrcorner \gamma) \otimes (e^i \wedge \delta) \right\rangle \\ &= \langle \alpha \otimes \beta, s^*(\gamma \otimes \delta) \rangle . \end{aligned}$$

□

Therefore, $s^*s \colon \Lambda^{p,q} \to \Lambda^{p,q}$ is self-adjoint, so it is diagonalizable. Noting that s is nilpotent because $s^{q+1} \colon \Lambda^{p,q} \to \Lambda^{p+q+1,-1} = 0$, for each eigenvector φ of s^*s, there exists an m such that $s^{m+1}\varphi = 0$ but $s^m\varphi \neq 0$. As we will prove in the following propositions, the eigenvalue corresponding to φ is uniquely determined by m, so m indexes the eigenspaces of s^*s.

Lemma 2.8. *If α is a k-covector, then*

$$\sum_{i=1}^n e^i \wedge (e_i \lrcorner \alpha) = k\alpha.$$

Proof. It suffices to check on a basis. If $\alpha = e^I$, then $e_i \lrcorner \alpha$ is nonzero if and only if $i \in I$. If so, then $e^i \wedge (e_i \lrcorner \alpha) = \alpha$. Therefore,

$$\sum_{i=1}^n e^i \wedge (e_i \lrcorner \alpha) = \sum_{i \in I} \alpha = k\alpha.$$

□

Proposition 2.9. *On $\Lambda^{p,q}$, we have*

$$ss^* - s^*s = p - q.$$

Proof. It suffices to check on simple tensors. We compute that

$$
\begin{aligned}
ss^*(\alpha\otimes\beta) &= \sum_{i,j}\left(e^i\wedge(e_j \lrcorner\, \alpha)\right)\otimes\left(e_i \lrcorner\,(e^j\wedge\beta)\right)\\
&= \sum_{i,j}\left(e^i\wedge(e_j \lrcorner\, \alpha)\right)\otimes\left(\delta_i^j\beta - e^j\wedge(e_i \lrcorner\, \beta)\right),\\
s^*s(\alpha\otimes\beta) &= \sum_{i,j}\left(e_j \lrcorner\,(e^i\wedge\alpha)\right)\otimes\left(e^j\wedge(e_i \lrcorner\, \beta)\right)\\
&= \sum_{i,j}\left(\delta_j^i\alpha - e^i\wedge(e_j \lrcorner\, \alpha)\right)\otimes\left(e^j\wedge(e_i \lrcorner\, \beta)\right),
\end{aligned}
$$

where δ denotes the Kronecker delta. Subtracting, we find that the second terms on each line cancel, leaving

$$
\begin{aligned}
(ss^*-s^*s)(\alpha\otimes\beta) &= \sum_{i,j}\left(e^i\wedge(e_j \lrcorner\, \alpha)\right)\otimes\delta_i^j\beta - \delta_j^i\alpha\otimes\left(e^j\wedge(e_i \lrcorner\, \beta)\right)\\
&= \sum_i (e^i\wedge(e_i \lrcorner\, \alpha))\otimes\beta - \alpha\otimes(e^i\wedge(e_i \lrcorner\, \beta))\\
&= p\alpha\otimes\beta - \alpha\otimes q\beta.
\end{aligned}
$$

□

Remark 2.10. Propositions 2.6, 2.7 and 2.9 also appear in [18, p. 55].

Proposition 2.9 gives us a quick way to determine when s is injective or surjective.

Lemma 2.11. *If $0\le q\le p\le n$, then $\ker s$ has a nonzero element. Likewise, if $0\le p\le q\le n$, then $\ker s^*$ has a nonzero element.*

Proof. Assume $0\le q\le p\le n$. Let $\alpha=\alpha_1\wedge\cdots\wedge\alpha_p$ be a nonzero p-covector, which is possible since $p\le n$. Since $q\le p$, we can let $\beta=\alpha_1\wedge\cdots\wedge\alpha_q$, so β is also nonzero and hence so is $\alpha\otimes\beta$. In the notation of Definition 2.4, we have $\beta_a=\alpha_a$ for $1\le a\le q$, which implies that $\beta_a\wedge\alpha=0$ for all a, so $s(\alpha\otimes\beta)=0$. The second claim follows by symmetry. □

Proposition 2.12. *Assume $0\le p,q\le n$, where $n=\dim V$. The operator $s\colon\Lambda^{p,q}\to\Lambda^{p+1,q-1}$ is injective if and only if $p<q$ and surjective if and only if $p\ge q-1$.*

Proof. Assume $p<q$. By Proposition 2.9, We have $s^*s=ss^*+q-p$. Since ss^* is positive semidefinite and $p<q$, we know that s^*s is positive definite, and hence s has zero kernel. Conversely, if $p\ge q$, then s has nonzero kernel by Lemma 2.11.

By symmetry, we have that $s^*\colon\Lambda^{p,q}\to\Lambda^{p-1,q+1}$ is injective if and only if $p>q$. Reindexing, we have that $s^*\colon\Lambda^{p+1,q-1}\to\Lambda^{p,q}$ is injective if and only if $p+1>q-1$. Hence its adjoint $s\colon\Lambda^{p,q}\to\Lambda^{p+1,q-1}$ is surjective if and only if $p+1>q-1$, which is equivalent to $p\ge q-1$. □

2.2. **The decomposition of double multicovectors.** We can naturally decompose the space of double multicovectors into the eigenspaces of s^*s. We begin by investigating the eigenvalues.

Lemma 2.13. *If $\varphi \in \Lambda^{p,q}$ is an eigenvector of s^*s with eigenvalue λ then $s\varphi$ is either zero or an eigenvector of s^*s with eigenvalue $\lambda + q - p - 2$.*

Proof. Since $s\varphi \in \Lambda^{p+1,q-1}$, we have by Proposition 2.9 that

$$(ss^* - s^*s)(s\varphi) = ((p+1) - (q-1))(s\varphi) = (p - q + 2)(s\varphi).$$

On the other hand, since $s^*s\varphi = \lambda\varphi$, we have

$$(ss^* - s^*s)(s\varphi) = ss^*s\varphi - s^*ss\varphi = s\lambda\varphi - s^*ss\varphi = (\lambda - s^*s)(s\varphi).$$

We conclude that

$$s^*s(s\varphi) = (\lambda + q - p - 2)(s\varphi)$$

as desired. □

Proposition 2.14. *If $\varphi \in \Lambda^{p,q}$ is an eigenvector of s^*s and m is the smallest integer such that $s^{m+1}\varphi = 0$, then φ has eigenvalue*

$$m(m + p - q + 1).$$

Proof. We induct on m. If $m = 0$ then $s\varphi = 0$ and so φ has eigenvalue 0, as desired.

Now assume that $m > 0$ and that the claim holds for $m - 1$ for all p and q. In particular, we can apply the claim to $s\varphi \in \Lambda^{p+1,q-1}$, because we know by the preceding lemma that $s\varphi$ is an eigenvector. So then, we have that $s\varphi$ has eigenvalue

$$\begin{aligned}(m-1)((m-1) + (p+1) - (q-1) + 1) &= (m-1)(m + p - q + 2) \\ &= m(m + p - q + 1) + q - p - 2.\end{aligned}$$

On the other hand, by the preceding lemma, if φ has eigenvalue λ, then $s\varphi$ has eigenvalue $\lambda + q - p - 2$, from which we conclude that $\lambda = m(m+p-q+1)$, as desired. □

Corollary 2.15. *If $\varphi \in \Lambda^{p,q}$ is an eigenvector of s^*s and m is the smallest integer such that $s^{m+1}\varphi = 0$, then $m \geq q - p$.*

Proof. The claim is obvious if $p \geq q$ because $m \geq 0$. On the other hand, if $p < q$, then s is injective by Proposition 2.12, so s^*s is positive definite, so the eigenvalue $m(m + p - q + 1)$ is positive. Since $m \geq 0$, we conclude that $m + p - q + 1$ is positive, which implies that $m \geq q - p$. □

Corollary 2.16. *For fixed p and q, the eigenvalues $m(m + p - q + 1)$ in the preceding proposition are strictly increasing in m. In particular, m is determined by the eigenvalue $m(m + p - q + 1)$.*

Proof. The claim follows because $m \geq 0$ and $m + p - q + 1 > 0$ and both are increasing in m. □

Definition 2.17. For an integer $m \geq 0$, let $\Lambda^{p,q}_m$ be the eigenspace of $s^*s\colon \Lambda^{p,q} \to \Lambda^{p,q}$ corresponding to eigenvalue $m(m+p-q+1)$. We define these spaces to be zero if $m < 0$.

Proposition 2.18. *We have the decomposition*

$$\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m.$$

Proof. The operator s^*s is self-adjoint and hence diagonalizable. Since s is nilpotent, for each eigenvector φ there exists a smallest integer m such that $s^{m+1}\varphi = 0$. We have shown that the corresponding eigenvalue is $m(m+p-q+1)$ and that this eigenvalue uniquely determines m. ☐

By symmetry, the above discussion works equally well if we swap the roles of p and q and consider the eigendecomposition of ss^* instead of s^*s.

Definition 2.19. For an integer $m^* \geq 0$, let ${}_{m^*}\Lambda^{p,q}$ be the eigenspace of $ss^*\colon \Lambda^{p,q} \to \Lambda^{p,q}$ corresponding to eigenvalue $m^*(m^*+q-p+1)$. We define these spaces to be zero if $m^* < 0$.

Proposition 2.20. *The dual decomposition is the same as the original decomposition with shifted index. Namely, ${}_{m^*}\Lambda^{p,q} = \Lambda^{p,q}_m$ for $m^* = m+p-q$.*

Proof. Say $\varphi \in {}_{m^*}\Lambda^{p,q}$, so φ is an eigenvector of ss^* with eigenvalue $m^*(m^*+q-p+1)$. By Proposition 2.9, we have that then φ is also an eigenvector of s^*s with eigenvalue $m^*(m^*+q-p+1)+q-p$. We compute that

$$\begin{aligned} m^*(m^*+q-p+1)+q-p &= (m+p-q)(m+1)+q-p \\ &= m(m+p-q+1), \end{aligned}$$

so hence $\varphi \in \Lambda^{p,q}_m$, as desired. ☐

Remark 2.21. We caution the reader that [12] uses the indexing for the dual decomposition. In other words, $\Lambda^{p,q}_m$ in [12] refers to ${}_m\Lambda^{p,q}$ in this paper.

2.3. **Properties of the decomposition.** Note that some terms of the decomposition may be zero. In the following propositions, we will determine exactly for which values of m the space $\Lambda^{p,q}_m$ is nonzero, as well as how the operators s and s^* interact with the decomposition.

Proposition 2.22. *The map s sends $\Lambda^{p,q}_m$ to $\Lambda^{p+1,q-1}_{m-1}$. Likewise, the map s^* sends $\Lambda^{p,q}_m$ to $\Lambda^{p-1,q+1}_{m+1}$.*

Proof. Let $\varphi \in \Lambda^{p,q}_m$. If $s\varphi$ is zero, the claim is tautological. Otherwise, φ is an eigenvector of s^*s and m is the smallest integer such that $s^{m+1}\varphi = 0$. By Lemma 2.13, $s\varphi$ is also an eigenvector of s^*s, and we have that $m-1$ is the smallest integer such that $s^{(m-1)+1}(s\varphi) = 0$. Hence, $s\varphi \in \Lambda^{p+1,q-1}_{m-1}$. So, s maps $\Lambda^{p,q}_m$ to $\Lambda^{p+1,q-1}_{m-1}$.

By symmetry, s^* maps ${}_{m^*}\Lambda^{p,q}$ to ${}_{m^*-1}\Lambda^{p-1,q+1}$. By Proposition 2.20, using $(m^*-1)-(p-1)+(q+1) = m+1$, we therefore have that s^* maps $\Lambda^{p,q}_m$ to $\Lambda^{p-1,q+1}_{m+1}$. ☐

Proposition 2.23. *The map $s\colon \Lambda^{p,q}_m \to \Lambda^{p+1,q-1}_{m-1}$ is injective if $m > 0$ and surjective if $m^* = m + p - q \geq 0$. Likewise, the map $s^*\colon \Lambda^{p,q}_m \to \Lambda^{p-1,q+1}_{m+1}$ is injective if $m^* > 0$ and surjective if $m \geq 0$.*

Proof. For $m > 0$, the map s^*s is a positive multiple of the identity on $\Lambda^{p,q}_m$, so $s\colon \Lambda^{p,q}_m \to \Lambda^{p+1,q-1}_{m-1}$ is injective and $s^*\colon \Lambda^{p+1,q-1}_{m-1} \to \Lambda^{p,q}_m$ is surjective. Reindexing, $s^*\colon \Lambda^{p,q}_m \to \Lambda^{p-1,q+1}_{m+1}$ is surjective for $m \geq 0$. By symmetry, we have that the operator $s^*\colon \Lambda^{p,q}_m \to \Lambda^{p-1,q+1}_{m+1}$ is injective if $m^* > 0$ and s is surjective if $m^* \geq 0$. □

Corollary 2.24. *Each of the maps $s\colon \Lambda^{p,q}_m \to \Lambda^{p+1,q-1}_{m-1}$ and $s^*\colon \Lambda^{p,q}_m \to \Lambda^{p-1,q+1}_{m+1}$ is an isomorphism if and only if the map's domain and codomain are both zero or both nonzero.*

Proof. We prove the claim for s, and the claim for s^* follows by symmetry. If both spaces are zero, the map is tautologically an isomorphism. If exactly one space is zero, the map cannot be an isomorphism. If both spaces are nonzero, then $\Lambda^{p,q}_m = {}_{m^*}\Lambda^{p,q}$ being nonzero implies $m^* \geq 0$ and $\Lambda^{p+1,q-1}_{m-1}$ being nonzero implies $m-1 \geq 0$, so s is an isomoprphism by Proposition 2.23. □

Corollary 2.25. *For a nonnegative integer l, the power $s^l\colon \Lambda^{p,q}_m \to \Lambda^{p+l,q-l}_{m-l}$ is surjective if $m^* = m+p-q \geq 0$. Likewise, the power $(s^*)^l\colon \Lambda^{p,q}_m \to \Lambda^{p-l,q+l}_{m+l}$ is surjective if $m \geq 0$.*

Proof. For the second statement, since $m, m+1, \dots, m+l-1$ are all nonnegative, each map in the composition

$$\Lambda^{p,q}_m \xrightarrow{s^*} \Lambda^{p-1,q+1}_{m+1} \xrightarrow{s^*} \dots \xrightarrow{s^*} \Lambda^{p-(l-1),q+l-1}_{m+l-1} \xrightarrow{s^*} \Lambda^{p-l,q+l}_{m+l}$$

is surjective, so the composition $(s^*)^l$ is surjective as well. The first statement follows by symmetry. □

Proposition 2.26. *The space $\Lambda^{p,q}_m$ is nonzero if and only if*

$$\max\{0, q-p\} \leq m \leq \min\{q, n-p\}, \tag{1}$$

where $n = \dim V$.

Proof. Assume that $\Lambda^{p,q}_m$ is nonzero, so $m \geq 0$. By Proposition 2.20, this space is equal to ${}_{m^*}\Lambda^{p,q}$, so $m^* \geq 0$ as well, which means that $m \geq q - p$. Recall that, for $\varphi \in \Lambda^{p,q}_m$, we have that m is the smallest integer such that $s^{m+1}\varphi = 0$. Observe that $s^{q+1}\colon \Lambda^{p,q} \to \Lambda^{p+q+1,-1} = 0$, so $s^{q+1}\varphi = 0$, and so $m \leq q$. Similarly, $s^{n-p+1}\colon \Lambda^{p,q} \to \Lambda^{n+1,p+q-n-1} = 0$, so $m \leq n - p$.

Now assume that Inequality (1) holds. We induct on m. Lemma 2.11 gives the base case because $\Lambda^{p,q}_0 = \ker s$ and $\Lambda^{p,q}_{q-p} = {}_0\Lambda^{p,q} = \ker s^*$. Now assume $m > 0$ and $m > q - p$, so $m^* > 0$. Thus, Proposition 2.23 tells us that s is an isomorphism between $\Lambda^{p,q}_m$ and $\Lambda^{p+1,q-1}_{m-1}$. By the inductive hypothesis, $\Lambda^{p+1,q-1}_{m-1}$ is nonzero if

$$\max\{0, (q-1)-(p+1)\} \leq m-1 \leq \min\{q-1, n-(p+1)\},$$

which simplifies to

$$\max\{1, q-p-1\} \le m \le \min\{q, n-p\}, \tag{2}$$

which is implied by Inequality (1) and the fact that $m > 0$. □

Remark 2.27. The preceding propositions yield a simple way to understand the injectivity and surjectivity of s from Proposition 2.12. On most summands of the decomposition, $s\colon \Lambda^{p,q}_m \to \Lambda^{p+1,q-1}_{m-1}$ is an isomorphism, but this fails exactly when one space is zero but the other is not.

To see when it is possible for exactly one of the spaces to be zero, we compare Inequalities (1) and (2). When can one inequality hold but the other fail? We see that (1) holds but (2) fails if and only if $m = 0 \ge q - p$, from which we conclude that s fails to be injective if and only if $p \ge q$. On the other hand, (2) holds but (1) fails if and only if $m = q - p - 1 \ge 1$, so s fails to be surjective if and only if $q \ge p + 2$.

In addition to providing a decomposition of $\Lambda^{p,q}$, the spaces $\Lambda^{p,q}_m$ also provide a decomposition of other subspaces of $\Lambda^{p,q}$ like the kernel and image of various powers of s and s^*.

Proposition 2.28. *For each nonnegative integer m, we have*

$$\ker s^m = \bigoplus_{l=0}^{m-1} \Lambda^{p,q}_l, \quad \operatorname{im}(s^*)^m = \bigoplus_{l=m}^{q} \Lambda^{p,q}_l, \quad \Lambda^{p,q}_m = \ker s^{m+1} \cap \operatorname{im}(s^*)^m.$$

Proof. Consider the space $\Lambda^{p-m,q+m}$, which we can decompose as

$$\Lambda^{p-m,q+m} = \bigoplus_{l=m}^{q+2m} \Lambda^{p-m,q+m}_{l-m}$$

by Proposition 2.26. If we apply $(s^*)^m$ to both sides, then we can use Corollary 2.25 to deduce that

$$(s^*)^m \Lambda^{p-m,q+m}_{l-m} = \begin{cases} \Lambda^{p,q}_l, & \text{if } m \le l \le q, \\ 0, & \text{if } l > q. \end{cases}$$

Thus,

$$\operatorname{im}(s^*)^m = \bigoplus_{l=m}^{q} \Lambda^{p,q}_l.$$

Taking the orthogonal complement of both sides yields $\ker s^m = \bigoplus_{l=0}^{m-1} \Lambda^{p,q}_l$, and from this it follows that $\Lambda^{p,q}_m = \ker s^{m+1} \cap \operatorname{im}(s^*)^m$. □

The preceding proposition can be used to determine when s^m is injective and when it is surjective, leading to the following generalization of Proposition 2.12.

Proposition 2.29. *Assume $0 \le p, q \le n$, where $n = \dim V$. Let m be a nonnegative integer. The operator $s^m : \Lambda^{p,q} \to \Lambda^{p+m,q-m}$ is injective if and only if $p < q - m + 1$ and surjective if and only if $p \ge q - m$.*

Proof. The operator s^m fails to be injective if and only if there is at least one nonzero summand in the decomposition $\ker s^m = \bigoplus_{l=0}^{m-1} \Lambda_l^{p,q}$. By Proposition 2.26, this happens if and only if there exists an $l \in \{0, 1, \ldots, m-1\}$ for which

$$\max\{0, q-p\} \le l \le \min\{q, n-p\},$$

i.e. the intervals $[0, m-1]$ and $[\max\{0, q-p\}, \min\{q, n-p\}]$ have nonempty intersection. This happens if and only if $\max\{0, q-p\} \le m-1$, which is equivalent to $p \ge q-m+1$. Therefore s^m is injective if and only if $p < q-m+1$.

To study surjectivity, we use the same strategy as in the proof of Proposition 2.12. By symmetry, we have that $(s^*)^m \colon \Lambda^{p,q} \to \Lambda^{p-m,q+m}$ is injective if and only if $q < p-m+1$. Reindexing, we have that $(s^*)^m \colon \Lambda^{p+m,q-m} \to \Lambda^{p,q}$ is injective if and only if $q-m < (p+m)-m+1$. Hence its adjoint $s^m \colon \Lambda^{p,q} \to \Lambda^{p+m,q-m}$ is surjective if and only if $q-m < p+1$, which is equivalent to $p \ge q-m$. □

Remark 2.30. Note that the definitions of the operators s and s^* do not require or depend on an inner product on V. Consequently, the eigendecomposition $\Lambda^{p,q} = \bigoplus \Lambda_m^{p,q}$ also does not depend on the inner product on V. (We did, however, use an arbitrary inner product to simplify the proofs of claims such as the diagonalizability of s^*s.)

2.4. Decomposition examples.

2.4.1. *The case $m = q$.* It turns out that the case $m = q$ is quite special. Let $k = p+q$. Observe that a k-covector, being antisymmetric in all indices, is, in particular, antisymmetric in the first p indices and in the last q indices. Thus, we have a natural inclusion $\Lambda^k \hookrightarrow \Lambda^{p,q}$. As we will see, $\Lambda_q^{p,q}$ is the image of this map. Conversely, the wedge product yields a natural map $\Lambda^{p,q} \to \Lambda^k$. As we will see, $\Lambda_q^{p,q}$ is the orthogonal complement of $\ker \wedge$.

Definition 2.31. For $k = p+q$, let

$$i^{p,q} \colon \Lambda^k \to \Lambda^{p,q}$$

denote the natural inclusion of antisymmetric k-tensors into the space of (p,q)-covectors given by.

$$(i^{p,q}\psi)(X_1, \ldots, X_p; Y_1, \ldots, Y_q) := \psi(X_1, \ldots, X_p, Y_1, \ldots, Y_q).$$

If either p or q are negative we define $i^{p,q}$ to be zero.

Definition 2.32. Let

$$\wedge \colon \Lambda^{p,q} \to \Lambda^k$$

denote the wedge product map defined on simple tensors by

$$\wedge(\alpha \otimes \beta) := \alpha \wedge \beta.$$

These operators have the following relationships with s.

Proposition 2.33. *We have*

$$si^{p,q} = (-1)^p(p+1)i^{p+1,q-1},$$
$$s^*i^{p,q} = (-1)^{p-1}(q+1)i^{p-1,q+1}.$$

Proof. Let $\psi \in \Lambda^k$. Using the antisymmetry of ψ, we have

$$\begin{aligned}
&(si^{p,q}\psi)(X_1,\dots,X_{p+1};Y_1,\dots,Y_{q-1})\\
&=\sum_{a=1}^{p+1}(-1)^{a-1}(i^{p,q}\psi)(X_1,\dots,\widehat{X_a},\dots,X_{p+1};X_a,Y_1,\dots,Y_{q-1})\\
&=\sum_{a=1}^{p+1}(-1)^{a-1}\psi(X_1,\dots,\widehat{X_a},\dots,X_{p+1},X_a,Y_1,\dots,Y_{q-1})\\
&=\sum_{a=1}^{p+1}(-1)^{p}\psi(X_1,\dots,X_a,\dots,X_{p+1},Y_1,\dots,Y_{q-1})\\
&=(-1)^p(p+1)\psi(X_1,\dots,X_a,\dots,X_{p+1},Y_1,\dots,Y_{q-1})
\end{aligned}$$

The second claim follows similarly, with care taken about the signs. □

Proposition 2.34. *On (p,q)-forms, we have*

$$\wedge\, s = (-1)^p q\,\wedge,$$
$$\wedge\, s^* = (-1)^{p-1} p\,\wedge.$$

Proof. On simple tensors, we have

$$\begin{aligned}
&\wedge\, s\big((\alpha_1\wedge\cdots\wedge\alpha_p)\otimes(\beta_1\wedge\cdots\wedge\beta_q)\big)\\
&=\sum_{a=1}^{q}(-1)^{a-1}\wedge\big((\beta_a\wedge\alpha_1\wedge\cdots\wedge\alpha_p)\otimes(\beta_1\wedge\cdots\wedge\widehat{\beta_a}\wedge\cdots\wedge\beta_q)\big)\\
&=\sum_{a=1}^{q}(-1)^{a-1}\beta_a\wedge\alpha_1\wedge\cdots\wedge\alpha_p\wedge\beta_1\wedge\cdots\wedge\widehat{\beta_a}\wedge\cdots\wedge\beta_q\\
&=\sum_{a=1}^{q}(-1)^{p}\alpha_1\wedge\cdots\wedge\alpha_p\wedge\beta_1\wedge\cdots\wedge\beta_a\wedge\cdots\wedge\beta_q\\
&=(-1)^p q(\alpha_1\wedge\cdots\wedge\alpha_p)\wedge(\beta_1\wedge\cdots\wedge\beta_q).
\end{aligned}$$

The second claim follows similarly, with care taken about the signs. □

Proposition 2.35. *The image of $i^{p,q}\colon \Lambda^k \to \Lambda^{p,q}$ is $\Lambda^{p,q}_q$.*

Proof. Assume $0 \le p, q$; otherwise, the claim is tautological because $\Lambda^{p,q} = 0$.

We induct on q. If $q = 0$, then $p = k$, and $m = 0$ is the only decomposition summand, so $i^{p,q}$ is just the isomorphism $\Lambda^k \to \Lambda^{k,0} = \Lambda^{k,0}_0$.

Now let $q \ge 1$ and assume that the claim holds for $q-1$. Then $i^{p,q} = (-1)^p q^{-1} s^* i^{p+1,q-1}$. By the inductive hypothesis, the image of $i^{p+1,q-1}$ is

$\Lambda^{p+1,q-1}_{q-1}$. Checking that $q-1 \geq 0$ and $(q-1)+(p+1)-(q-1) = p+1 > 0$, Proposition 2.23 tells us that s^* is an isomorphism from $\Lambda^{p+1,q-1}_{q-1}$ to $\Lambda^{p,q}_q$. Hence, the image of $i^{p,q} = (-1)^p q^{-1} s^* i^{p+1,q-1}$ is $\Lambda^{p,q}_q$. □

Proposition 2.36. *The space* $\Lambda^{p,q}_q$ *is the orthogonal complement of the kernel of* $\wedge\colon \Lambda^{p,q} \to \Lambda^k$.

Proof. Again, assume $0 \leq p, q$; otherwise, the claim is tautological.

We induct on q on the statement that $\wedge\colon \Lambda^{p,q}_m \to \Lambda^k$ is zero if $m < q$ and is an isomorphism if $m = q$. As before, if $q = 0$, then $p = k$, and $m = 0$ is the only decomposition summand, so we see that $\wedge\colon \Lambda^{k,0}_0 = \Lambda^{k,0} \to \Lambda^k$ is the obvious isomorphism.

Now assume $q \geq 1$ and that the proposition holds for $q-1$. On (p,q)-forms, we have $\wedge = (-1)^p q^{-1} \wedge s$.

Consider first the case $m = q$. Since $m > 0$ and $m^* = m+p-q = p \geq 0$, Proposition 2.23 tells us that s is an isomorphism from $\Lambda^{p,q}_q$ to $\Lambda^{p+1,q-1}_{q-1}$, and then $\wedge$ is an isomorphism from $\Lambda^{p+1,q-1}_{q-1}$ to Λ^k by the inductive hypothesis. Hence the composition $\wedge = (-1)^p q^{-1} \wedge s$ is an isomorphism from $\Lambda^{p,q}_q$ to Λ^k.

Now consider the case $m < q$. If the space $\Lambda^{p,q}_m$ is zero, then the claim is tautological, so we may assume $m \geq 0$ and $m^* \geq 0$. If $m = 0$, then s sends $\Lambda^{p,q}_m$ to zero. If $m > 0$, then s is an isomorphism from $\Lambda^{p,q}_m$ to $\Lambda^{p+1,q-1}_{m-1}$ by Proposition 2.23, and then $\wedge$ sends $\Lambda^{p+1,q-1}_{m-1}$ to zero by the inductive hypothesis. Either way, the composition $\wedge = (-1)^p q^{-1} \wedge s$ is zero, as desired. □

2.4.2. *The case* $(p,q) = (1,1)$. When $p = q = 1$, the decomposition in Proposition 2.18 reads
$$\Lambda^{1,1} = \Lambda^{1,1}_0 \oplus \Lambda^{1,1}_1.$$
As shown in Proposition 2.35, $\Lambda^{1,1}_1$ is the image of Λ^2 under the natural inclusion $i^{1,1} : \Lambda^2 \hookrightarrow \Lambda^{1,1}$. In other words, $\Lambda^{1,1}_1$ consists of antisymmetric bilinear forms. Consequently, $\Lambda^{1,1}_0$ consists of symmetric bilinear forms.

2.4.3. *The case* $(p,q) = (2,1)$. When $(p,q) = (2,1)$, the decomposition in Proposition 2.18 reads
$$\Lambda^{2,1} = \Lambda^{2,1}_0 \oplus \Lambda^{2,1}_1.$$
In dimension $n = 3$, we can understand these spaces by identifying elements of $\Lambda^{2,1}$ with matrices. Specifically, we can write any $\varphi \in \Lambda^{2,1}$ in the form
$$\varphi = \sum_{i,j=1}^{3} a_{ij} \alpha^i \otimes e^j,$$
where e^1, e^2, e^3 is a basis for V^*, $\alpha^1 = e^2 \wedge e^3$, $\alpha^2 = e^3 \wedge e^1$, and $\alpha^3 = e^1 \wedge e^2$. In this basis,
$$s\varphi = \sum_{i,j=1}^{3} a_{ij} e^j \wedge \alpha^i = \left(\sum_{i=1}^{3} a_{ii} \right) e^1 \wedge e^2 \wedge e^3,$$

so φ belongs to $\Lambda_0^{2,1} = \ker s$ if and only if the matrix of coefficients $[a_{ij}]_{i,j=1}^3$ is trace-free. Thus, in three dimensions, the decomposition $\Lambda^{2,1} = \Lambda_0^{2,1} \oplus \Lambda_1^{2,1}$ is simply the decomposition of a 3×3 matrix into its deviatoric part plus a multiple of the identity.

2.4.4. *The case* $(p,q) = (2,2)$. When $(p,q) = (2,2)$, there are three summands in the decomposition:

$$\Lambda^{2,2} = \Lambda_0^{2,2} \oplus \Lambda_1^{2,2} \oplus \Lambda_2^{2,2}.$$

We will first discuss the summands in any dimension n and then specialize to $n = 3$.

The space $\Lambda_0^{2,2} = \ker s$ consists of $(2,2)$-forms that satisfy the *Bianchi identity*

$$\begin{aligned}(s\varphi)(X,Y,Z;W) &= \varphi(Y,Z;X,W) - \varphi(X,Z;Y,W) + \varphi(X,Y;Z,W) \\ &= \varphi(Y,Z;X,W) + \varphi(Z,X;Y,W) + \varphi(X,Y;Z,W) = 0.\end{aligned}$$

Such a $(2,2)$-form is called an *algebraic curvature tensor* because it possesses the same symmetries as the Riemann curvature tensor. Namely, $\varphi(X,Y;Z,W)$ alternates in X and Y, alternates in Z and W, and satisfies the Bianchi identity. It can be shown [20, p. 204] that such tensors automatically possess the additional symmetry

$$\varphi(X,Y;Z,W) = \varphi(Z,W;X,Y).$$

By Proposition 2.35, the space $\Lambda_2^{2,2}$ is the image of Λ^4 under the natural inclusion $i^{2,2} : \Lambda^4 \hookrightarrow \Lambda^{2,2}$. As such, it consists of tensors that alternate in all 4 arguments. In particular, such tensors satisfy the symmetry $\varphi(X,Y;Z,W) = \varphi(Z,W;X,Y)$ as well. This implies that any $(2,2)$-form satisfying the *skew*-symmetry

$$\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$$

must belong to the remaining space $\Lambda_1^{2,2}$. In fact, $\Lambda_1^{2,2}$ consists precisely of those $(2,2)$-forms φ satisfying $\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$. One way to show this is to count dimensions: By Proposition 2.23, the dimension of $\Lambda_1^{2,2}$ matches the dimension of $\Lambda_0^{3,1}$, and this space is the kernel of the surjective map $s : \Lambda^{3,1} \to \Lambda^{4,0}$. Hence it has dimension

$$\dim \Lambda^{3,1} - \dim \Lambda^{4,0} = n\binom{n}{3} - \binom{n}{4} = \frac{1}{2}\binom{n}{2}\left(\binom{n}{2} - 1\right).$$

Since this number matches the dimension of the space of $(2,2)$-forms satisfying the skew-symmetry $\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$, the claim follows.

In dimension $n = 3$, the situation simplifies. There are no 4-forms in 3 dimensions, so $\Lambda_2^{2,2}$ vanishes. By the discussion above, the remaining spaces $\Lambda_0^{2,2}$ and $\Lambda_1^{2,2}$ must therefore consist of all symmetric $(2,2)$-forms and all skew-symmetric $(2,2)$-forms, respectively. If, in the notation of Section 2.4.3, we identify a $(2,2)$-form $\varphi = \sum_{i,j=1}^3 a_{ij}\alpha^i \otimes \alpha^j$ with a 3×3 matrix

$A = [a_{ij}]_{i,j=1}^3$, then φ belongs to $\Lambda_0^{2,2}$ (respectively, $\Lambda_1^{2,2}$) if and only if A is symmetric (respectively, skew-symmetric).

2.5. **Additional operations on double multicovectors.** In addition to s and s^*, there are several other natural operations on double multicovectors. As before, we work on a vector space V of dimension n.

Definition 2.37. Let the *transposition operator*

$$\tau : \Lambda^{p,q} \to \Lambda^{q,p}$$

be the involution that swaps the two factors, that is, on simple tensors, we have

$$\tau(\alpha \otimes \beta) = \beta \otimes \alpha.$$

Definition 2.38. Let the *double wedge product*, sometimes called the Kulkarni–Nomizu product, be the binary operation

$$\owedge : \Lambda^{p,q} \times \Lambda^{p',q'} \to \Lambda^{p+p',q+q'},$$

that is defined on simple tensors by

$$(\alpha \otimes \beta) \owedge (\gamma \otimes \delta) = (\alpha \wedge \gamma) \otimes (\beta \wedge \delta).$$

Definition 2.39. Let the *double Hodge star* be the operator

$$\circledast : \Lambda^{p,q} \to \Lambda^{n-p,n-q}$$

that is defined on simple tensors by

$$\circledast(\alpha \otimes \beta) = \star\alpha \otimes \star\beta,$$

where $\star$ is the Hodge star.

Remark 2.40. Similarly to s and s^*, the definitions of the operators τ and $\owedge$ do not require or depend on an inner product on V. In contrast, because $\star$ does depend on an inner product on V, so does $\circledast$.

Our goal now is to prove the compatibility of the double Hodge star with the above decomposition of double multicovectors. We begin with some basic properties of these operators.

Proposition 2.41. *We have*

$$\tau s = s^* \tau,$$
$$\tau s^* = s \tau,$$
$$\tau \circledast = \circledast \tau.$$

Proof. The claims follow from the symmetry between the definitions of s and s^*, and from the symmetry in the definition of $\circledast$. □

By symmetry, τ sends the decomposition to the dual decomposition.

Proposition 2.42. *The transposition τ is an isomorphism between $\Lambda_m^{p,q}$ and $\Lambda_{m^*}^{q,p}$, where $m^* = m + p - q$.*

Proof. Say $\varphi \in \Lambda^{q,p}_{m^*}$. Then, by Definition 2.17, φ is an eigenvector of s^*s with eigenvalue $m^*(m^*+q-p+1)$. Since $\tau s^* s = s s^* \tau$, we have that $\tau\varphi \in \Lambda^{p,q}$ is an eigenvector of ss^* with the same eigenvalue, $m^*(m^*+q-p+1)$. Therefore, by Definition 2.19 and Proposition 2.20, $\tau\varphi \in {}_{m^*}\Lambda^{p,q} = \Lambda^{p,q}_m$, as desired. □

Proposition 2.43. *On $\Lambda^{p,q}$, we have*

$$\circledast^2 = (-1)^{p(n-p)+q(n-q)}.$$

Proof. The claim follows from the fact that, on k-covectors, $\star^2 = (-1)^{k(n-k)}$. □

Proposition 2.44. *For any $\varphi, \psi \in \Lambda^{p,q}$, we have*

$$\langle \varphi, \psi \rangle = \circledast^{-1}(\varphi \owedge \circledast \psi).$$

Proof. It suffices to prove the claim for simple tensors $\varphi = \alpha \otimes \beta$ and $\psi = \gamma \otimes \delta$. By properties of the Hodge star, we have

$$\begin{aligned}\langle \varphi, \psi \rangle = \langle \alpha, \gamma \rangle \langle \beta, \delta \rangle &= \left(\star^{-1}(\alpha \wedge \star\gamma)\right)\left(\star^{-1}(\beta \wedge \star\delta)\right) \\ &= \circledast^{-1}\left((\alpha \otimes \beta) \owedge \circledast(\gamma \otimes \delta)\right).\end{aligned}$$ □

Lemma 2.45. *For any $\varphi \in \Lambda^{p,q}$ and $\psi \in \Lambda^{p',q'}$, we have*

$$s(\varphi \owedge \psi) = (s\varphi) \owedge \psi + (-1)^k \varphi \owedge (s\psi),$$

where $k = p + q$.

Proof. It is not hard to verify this claim using Proposition 2.6 and properties of the interior product. See also [15, Proposition 2.1]. □

Proposition 2.46. *On $\Lambda^{p,q}$, we have*

$$\begin{aligned}\circledast s^* &= (-1)^{k+1} s \circledast, \\ \circledast s &= (-1)^{k+1} s^* \circledast,\end{aligned}$$

where $k = p + q$.

Proof. Let $\varphi \in \Lambda^{p-1,q+1}$ and $\psi \in \Lambda^{p,q}$. Notice that $\varphi \owedge \circledast \psi$ belongs to $\Lambda^{n-1,n+1} = 0$, so $s(\varphi \owedge \circledast \psi) = 0$. Note also that $(p-1)+(q+1) = k$. Thus, Lemma 2.45 implies that

$$(s\varphi) \owedge \circledast \psi = (-1)^{k+1} \varphi \wedge (s \circledast \psi).$$

Equivalently,

$$\langle s\varphi, \psi \rangle = (-1)^{k+1} \langle \varphi, \circledast^{-1} s \circledast \psi \rangle.$$

Since φ and ψ are arbitrary and s and s^* are adjoints, we conclude that

$$s^* = (-1)^{k+1} \circledast^{-1} s \circledast,$$

from which the first claim follows. Conjugating by τ and using the fact that τ preserves k and commutes with $\circledast$ and hence $\circledast^{-1}$, we obtain

$$\tau s^* \tau = (-1)^{k+1} \circledast^{-1} \tau s \tau \circledast,$$

and so the second claim follows by $\tau s^* \tau = s$ and $\tau s \tau = s^*$. □

Proposition 2.47. *The operators s^*s and $\tau\circledast$ commute.*

Proof. On $\Lambda^{p,q}$, with $k = p+q$, noting that s preserves k, we compute that

$$s^*s\tau\circledast = s^*\tau s^*\circledast = \tau s s^*\circledast = (-1)^{k+1}\tau s\circledast s = \tau\circledast s^*s. \qquad \square$$

Proposition 2.48. *The isomorphism $\tau\circledast$ preserves the decomposition of double multicovectors, sending $\Lambda^{p,q}_m$ to $\Lambda^{n-q,n-p}_m$.*

Proof. Let $\varphi \in \Lambda^{p,q}$ be an eigenvector of s^*s. Then $\tau\circledast\varphi \in \Lambda^{n-q,n-p}$ is an eigenvector of s^*s with the same eigenvalue. We then observe that

$$m(m+p-q+1) = m(m+(n-q)-(n-p)+1),$$

so this eigenvalue corresponds to the same value of m in both $\Lambda^{p,q}$ and $\Lambda^{n-q,n-p}$. $\square$

We can conclude that $\circledast$ by itself sends the decomposition to the dual decomposition.

Proposition 2.49. *The isomorphism $\circledast$ sends $\Lambda^{p,q}_m$ to $\Lambda^{n-p,n-q}_{m^*}$, where $m^* = m+p-q$.*

Proof. Since τ is an involution that commutes with $\circledast$, we have $\circledast = (\tau\circledast)\tau$. By Proposition 2.42, τ sends $\Lambda^{p,q}_m$ to $\Lambda^{q,p}_{m^*}$. By Proposition 2.48, $\tau\circledast$ sends $\Lambda^{q,p}_{m^*}$ to $\Lambda^{n-p,n-q}_{m^*}$. $\square$

3. Double forms

Definition 3.1. Let M be a smooth manifold. For $p+q=k$, let the space of (p,q)-*forms* or *double forms*, denoted $\Lambda^{p,q}(M)$, be the space of smooth covariant k-tensor fields on M that are antisymmetric in the first p indices and antisymmetric in the last q indices. In other words, $\Lambda^{p,q}(M)$ is the space of smooth sections of the bundle $\bigwedge^p T^*M \otimes \bigwedge^q T^*M$.

Note that, at each point $x \in M$, this bundle gives the vector space $\bigwedge^p T^*_xM \otimes \bigwedge^q T^*_xM$, so we just have the constructions from the previous subsections with $V = T_xM$. Consequently, the operators on double multicovectors from the previous section can be applied pointwise to yield operators on double forms.

Definition 3.2. We define the operators

$$\begin{aligned} s\colon \Lambda^{p,q}(M) &\to \Lambda^{p+1,q-1}(M),\\ s^*\colon \Lambda^{p,q}(M) &\to \Lambda^{p-1,q+1}(M),\\ \tau\colon \Lambda^{p,q}(M) &\to \Lambda^{q,p}(M),\\ \circledast\colon \Lambda^{p,q}(M) &\to \Lambda^{n-p,n-q}(M), \end{aligned}$$

by applying the corresponding double multicovector operators pointwise. Here $n = \dim M$, and $\circledast$ requires and depends on a Riemannian metric on M.

The preceding formulas relating these operators on double multicovectors apply equally well to double forms, and, likewise, double forms have the same eigendecomposition.

Proposition 3.3. *We have the decomposition*

$$\Lambda^{p,q}(M) = \bigoplus_m \Lambda^{p,q}_m(M),$$

where $\max\{0, q-p\} \leq m \leq \min\{q, n-p\}$ *and* $\Lambda^{p,q}_m(M)$ *is the space of eigenfunctions of* s^*s *with eigenvalue* $m(m+p-q+1)$.

Except for $\circledast$, which depends on a Riemannian metric, these operators commute with pullback by smooth maps.

Proposition 3.4. *Let* $\Phi\colon M \to N$ *be a smooth map between smooth manifolds. Then the pullback map* $\Phi^*\colon \Lambda^{p,q}(N) \to \Lambda^{p,q}(M)$ *commutes with* s, s^*, *and* τ, *and* Φ^* *respects the decomposition, sending* $\Lambda^{p,q}_m(N)$ *to* $\Lambda^{p,q}_m(M)$.

Proof. Because the wedge product commutes with pullback, both $s\Phi^*$ and Φ^*s, when applied to $(\alpha_1 \wedge \cdots \wedge \alpha_p) \otimes (\beta_1 \wedge \cdots \wedge \beta_q)$, are equal to

$$\sum_{a=1}^{q} (-1)^{a-1} (\Phi^*\beta_a \wedge \Phi^*\alpha_1 \wedge \cdots \wedge \Phi^*\alpha_p) \otimes (\Phi^*\beta_1 \wedge \cdots \wedge \widehat{\Phi^*\beta_a} \wedge \cdots \wedge \Phi^*\beta_q).$$

We can similarly show that Φ^* commutes with s^* and τ. Consequently, Φ^* commutes with s^*s, and so Φ^* respects the eigendecomposition of s^*s, with the same eigenvalues (possibly sending some eigenvectors to zero). □

If M has a Riemannian metric (or simply a connection ∇ on the tangent bundle), then we can define the exterior covariant derivative on $\Lambda^{p,q}(M)$ in two different ways, since we can view (p,q)-forms as Λ^p-valued q-forms or Λ^q-valued p-forms. However, we will only need these operators when M is simply Euclidean space, so instead we present the definition in this specialized context.

3.1. **Double forms on Euclidean space.**

In this subsection, we will have M be $\mathbb{R}^{n+1}$, with coordinates $(x^0, \dots, x^n)$. Note that the dimension here is $n+1$, which differs from the convention in the earlier subsections.

Notation 3.5. When there is no risk of confusion, we will let Λ^k and $\Lambda^{p,q}$ denote $\Lambda^k(\mathbb{R}^{n+1})$ and $\Lambda^{p,q}(\mathbb{R}^{n+1})$, respectively. For a multi-index $I = (i_1, \dots, i_k)$, let $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k$, and let $dx^{I,J} = dx^I \otimes dx^J \in \Lambda^{p,q}$.

Definition 3.6. We define natural operators

$$d_L\colon \Lambda^{p,q} \to \Lambda^{p+1,q}, \qquad d_R\colon \Lambda^{p,q} \to \Lambda^{p,q+1},$$

by

$$d_L(f\,dx^{I,J}) = \left(df \wedge dx^I\right) \otimes dx^J, \qquad d_R(f\,dx^{I,J}) = dx^I \otimes \left(df \wedge dx^J\right).$$

Here, f is an arbitrary smooth function, and we extend these definitions by linearity.

Proposition 3.7. *The operators d_L and d_R commute.*

Proof. Applying both $d_L d_R$ and $d_R d_L$ to $f\,dx^{I,J}$, by the symmetry of the Hessian, we obtain

$$\sum_{i,j} \frac{\partial f}{\partial x^i \partial x^j} \left(dx^i \wedge dx^I\right) \otimes \left(dx^j \wedge dx^J\right). \qquad \square$$

Definition 3.8. The *tautological vector field* is

$$X_{\mathrm{id}} := \sum_{i=0}^{n} x^i \frac{\partial}{\partial x^i}.$$

If α is a k-form, we let the *Koszul operator* κ denote contraction with X_{id}, that is,

$$\kappa\alpha := X_{\mathrm{id}} \lrcorner\, \alpha.$$

For a double form, we can apply κ to either the left factor or the right factor; we denote these operators by κ_L and κ_R, respectively. Namely, we have,

$$\begin{aligned} \kappa_L \colon \Lambda^{p,q} &\to \Lambda^{p-1,q}, & \kappa_R \colon \Lambda^{p,q} &\to \Lambda^{p,q-1}, \\ \alpha \otimes \beta &\mapsto (\kappa\alpha) \otimes \beta, & \alpha \otimes \beta &\mapsto \alpha \otimes (\kappa\beta). \end{aligned}$$

Proposition 3.9. *The operators κ_L and κ_R commute.*

Proof. Applying both $\kappa_L\kappa_R$ and $\kappa_R\kappa_L$ to $\alpha \otimes \beta$, we obtain

$$(X_{\mathrm{id}} \lrcorner\, \alpha) \otimes (X_{\mathrm{id}} \lrcorner\, \beta). \qquad \square$$

There are also several nontrivial commutator relationships between our operators. The first one that we prove below is a special case of a more general relationship that appears in [18, p. 55] and [15, p. 259].

Proposition 3.10. *We have*

$$\begin{aligned} \kappa_L s + s\kappa_L &= \kappa_R, & \kappa_R s^* + s^*\kappa_R &= \kappa_L \\ \kappa_L s^* + s^*\kappa_L &= 0, & \kappa_R s + s\kappa_R &= 0. \end{aligned}$$

Proof. Using Proposition 2.6, we have

$$\kappa_L s(\alpha \otimes \beta) = \sum_{i,j} x^j \left(\frac{\partial}{\partial x^j} \lrcorner\, (dx^i \wedge \alpha)\right) \otimes \left(\frac{\partial}{\partial x^i} \lrcorner\, \beta\right),$$

$$s\kappa_L(\alpha \otimes \beta) = \sum_{i,j} x^j \left(dx^i \wedge \left(\frac{\partial}{\partial x^j} \lrcorner\, \alpha\right)\right) \otimes \left(\frac{\partial}{\partial x^i} \lrcorner\, \beta\right).$$

Adding, we obtain

$$\begin{aligned} (\kappa_L s + s\kappa_L)(\alpha \otimes \beta) &= \sum_{i,j} x^j \left(\frac{\partial x^i}{\partial x^j}\alpha\right) \otimes \left(\frac{\partial}{\partial x^i} \lrcorner\, \beta\right) \\ &= \sum_i x^i \alpha \otimes \left(\frac{\partial}{\partial x^i} \lrcorner\, \beta\right) \\ &= \alpha \otimes (X_{\mathrm{id}} \lrcorner\, \beta), \end{aligned}$$

as desired.

Meanwhile,

$$\kappa_L s^*(\alpha \otimes \beta) = \sum_{i,j} x^j \left(\frac{\partial}{\partial x^j} \lrcorner \left(\frac{\partial}{\partial x^i} \lrcorner \alpha \right) \right) \otimes \left(dx^i \wedge \beta \right),$$

$$s^* \kappa_L(\alpha \otimes \beta) = \sum_{i,j} x^j \left(\frac{\partial}{\partial x^i} \lrcorner \left(\frac{\partial}{\partial x^j} \lrcorner \alpha \right) \right) \otimes \left(dx^i \wedge \beta \right).$$

The sum is zero by antisymmetry of contraction.

The remaining two claims follows by symmetry. □

We can now show that the operator $\kappa_L \kappa_R = \kappa_R \kappa_L$ respects the decomposition $\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m$.

Proposition 3.11. *The operator $\kappa_L \kappa_R = \kappa_R \kappa_L$ commutes with $s^* s$.*

Proof. Using the fact that $\kappa_L^2 = \kappa_R^2 = 0$, we compute

$$\begin{aligned} s^* s \kappa_L \kappa_R = s^*(\kappa_R - \kappa_L s)\kappa_R = -s^* \kappa_L s \kappa_R \\ = -\kappa_L s^* \kappa_R s = -\kappa_L(\kappa_L - \kappa_R s^*)s = \kappa_L \kappa_R s^* s, \end{aligned}$$

as desired. □

Proposition 3.12. *The operator $\kappa_L \kappa_R \colon \Lambda^{p,q} \to \Lambda^{p-1,q-1}$ sends $\Lambda^{p,q}_m$ to $\Lambda^{p-1,q-1}_m$.*

Proof. If φ is an eigenvalue of $s^* s$ with eigenvalue $m(m+p-q+1)$, then, by the above proposition,

$$s^* s(\kappa_L \kappa_R \varphi) = m(m+p-q+1)(\kappa_L \kappa_R \varphi).$$

Thus, $\kappa_L \kappa_R \varphi$ is either zero or an eigenvalue of $s^* s$ with eigenvalue $m(m+p-q+1)$. If $\kappa_L \kappa_R \varphi$ is zero, then it is in $\Lambda^{p-1,q-1}_m$ tautologically; if it is nonzero, then it is in $\Lambda^{p-1,q-1}_m$ because $m(m+(p-1)-(q-1)+1) = m(m+p-q+1)$. □

We can prove an analogous result for the operator $d_L d_R$.

Proposition 3.13. *We have*

$$d_L s + s d_L = 0, \qquad d_R s^* + s^* d_R = 0$$
$$d_L s^* + s^* d_L = d_R, \qquad d_R s + s d_R = d_L.$$

Proof. Using Proposition 2.6, we have

$$d_L s(f\, dx^{I,J}) = \sum_{i,j} \frac{\partial f}{\partial x^j}(dx^j \wedge dx^i \wedge dx^I) \otimes \left(\frac{\partial}{\partial x^i} \lrcorner dx^J \right),$$

$$s d_L(f\, dx^{I,J}) = \sum_{i,j} \frac{\partial f}{\partial x^j}(dx^i \wedge dx^j \wedge dx^I) \otimes \left(\frac{\partial}{\partial x^i} \lrcorner dx^J \right).$$

The sum is zero by the antisymmetry of wedge.

Meanwhile,

$$d_L s^*(f\,dx^{I,J}) = \sum_{i,j} \frac{\partial f}{\partial x^j}\left(dx^j \wedge \left(\frac{\partial}{\partial x^i} \lrcorner\, dx^I\right)\right) \otimes (dx^i \wedge dx^J),$$

$$s^* d_L(f\,dx^{I,J}) = \sum_{i,j} \frac{\partial f}{\partial x^j}\left(\frac{\partial}{\partial x^i} \lrcorner\, (dx^j \wedge dx^I)\right) \otimes (dx^i \wedge dx^J).$$

Adding, we obtain

$$\begin{aligned}(d_L s^* + s^* d_L)(f\,dx^{I,J}) &= \sum_{i,j} \frac{\partial f}{\partial x^j}\left(\frac{\partial x^j}{\partial x^i}\,dx^I\right) \otimes (dx^i \wedge dx^J) \\ &= \sum_i \frac{\partial f}{\partial x^i}\,dx^I \wedge (dx^i \wedge dx^J) \\ &= d_R(f\,dx^{I,J}).\end{aligned}$$

The remaining claims follow by symmetry. □

Proposition 3.14. *The operator $d_L d_R$ commutes with s^*s.*

Proof. We have

$$\begin{aligned}d_L d_R s^* s = -d_L s^* d_R s = -(d_R - s^* d_L) d_R s = s^* d_L d_R s \\ = s^* d_L(d_L - s d_R) = -s^* d_L s d_R = s^* s d_L d_R\end{aligned}$$

□

Proposition 3.15. *The operator $d_L d_R \colon \Lambda^{p,q} \to \Lambda^{p+1,q+1}$ sends $\Lambda^{p,q}_m$ to $\Lambda^{p+1,q+1}_m$.*

Proof. The proof is analogous to the proof of Proposition 3.12. □

Finally, we have commutation relations between the exterior derivatives and the Koszul operators.

Proposition 3.16. *We have*

$$d_L \kappa_R - \kappa_R d_L = s, \qquad d_R \kappa_L - \kappa_L d_R = s^*.$$

Proof. We have

$$\begin{aligned}d_L \kappa_R \left(f\,dx^{I,J}\right) &= d_L \sum_i x^i f\,dx^I \otimes \left(\frac{\partial}{\partial x^i} \lrcorner\, dx^J\right) \\ &= \sum_i \left(\left(f\,dx^i + x^i\,df\right) \wedge dx^I\right) \otimes \left(\frac{\partial}{\partial x^i} \lrcorner\, dx^I\right).\end{aligned}$$

$$\kappa_R d_L \left(f\,dx^{I,J}\right) = \sum_i \left(x^i\,df \wedge dx^I\right) \otimes \left(\frac{\partial}{\partial x^i} \lrcorner\, dx^J\right).$$

Subtracting, we obtain $s\left(f\,dx^{I,J}\right)$ using Proposition 2.6. The second equation follows by symmetry. □

3.2. **Polynomial double forms on Euclidean space.**

Definition 3.17. Let $\mathcal{H}_r\Lambda^{p,q}(\mathbb{R}^{n+1})$ or simply $\mathcal{H}_r\Lambda^{p,q}$ denote the space of double forms with homogeneous polynomial coefficients of degree r. In other words, $\mathcal{H}_r\Lambda^{p,q}$ is spanned by $f\,dx^{I,J}$, where f is a homogeneous polynomial of degree r. We will define the decomposition component $\mathcal{H}_r\Lambda^{p,q}_m$ similarly, and we will also occasionally need analogously defined spaces $\mathcal{H}_r\Lambda^k$ of k-forms, as well as the space $\mathcal{H}_r$ of scalar fields, which is simply the space of homogeneous polynomials of degree r.

Observe that $\kappa_L\kappa_R\colon \mathcal{H}_{r-2}\Lambda^{p+1,q+1} \to \mathcal{H}_r\Lambda^{p,q}$. The image of this map will be important enough to merit a definition.

Definition 3.18. Let

$$\mathcal{H}_r^-\Lambda^{p,q} := \kappa_L\kappa_R\mathcal{H}_{r-2}\Lambda^{p+1,q+1}.$$

We likewise let

$$\mathcal{H}_r^-\Lambda^{p,q}_m := \kappa_L\kappa_R\mathcal{H}_{r-2}\Lambda^{p+1,q+1}_m.$$

Note that $\mathcal{H}_r^-\Lambda^{p,q} = 0$ if $r < 2$. As the notation suggests, $\mathcal{H}_r^-\Lambda^{p,q}_m$ is a subspace of $\mathcal{H}_r\Lambda^{p,q}_m$ by Proposition 3.12. Specifically, we have the following.

Proposition 3.19. *We have*

$$\mathcal{H}_r^-\Lambda^{p,q}_m = \mathcal{H}_r^-\Lambda^{p,q} \cap \Lambda^{p,q}_m.$$

Proof. If $\varphi \in \kappa_L\kappa_R\mathcal{H}_{r-2}\Lambda^{p+1,q+1}_m$, then it is in $\mathcal{H}_r^-\Lambda^{p,q}$ by definition and in $\Lambda^{p,q}_m$ by Proposition 3.12.

Conversely, assume that $\varphi \in \mathcal{H}_r^-\Lambda^{p,q} \cap \Lambda^{p,q}_m$. By definition, $\varphi = \kappa_L\kappa_R\psi$ for some $\psi \in \mathcal{H}_{r-2}\Lambda^{p+1,q+1}$, but ψ might not be in the decomposition summand $\Lambda^{p+1,q+1}_m$. However, we can decompose $\psi = \sum_{m'}\psi_{m'}$ where each $\psi_{m'} \in \Lambda^{p+1,q+1}_{m'}$. The polynomial coefficients are unaffected by the decomposition, so, in fact, $\psi_{m'} \in \mathcal{H}_{r-2}\Lambda^{p+1,q+1}_{m'}$. Letting $\varphi_{m'} = \kappa_L\kappa_R\psi_{m'}$, we have that $\varphi = \sum_{m'}\varphi_{m'}$. By Proposition 3.12, $\varphi_{m'} \in \Lambda^{p,q}_{m'}$. Since $\varphi \in \Lambda^{p,q}_m$, we conclude that $\varphi_{m'} = 0$ unless $m = m'$, so $\varphi = \varphi_m = \kappa_L\kappa_R\psi_m$, so $\varphi \in \mathcal{H}_r^-\Lambda^{p,q}_m$ by definition. □

Since κ_L commutes with κ_R and $\kappa_L^2 = \kappa_R^2 = 0$, we see that anything in $\mathcal{H}_r^-\Lambda^{p,q}$ is in the kernel of both κ_L and κ_R. Through the next few propositions, we will see that this condition almost characterizes $\mathcal{H}_r^-\Lambda^{p,q}$.

Proposition 3.20. *On $\mathcal{H}_r\Lambda^{p,q}$, we have*

$$d_L\kappa_L + \kappa_L d_L = r + p, \qquad d_R\kappa_R + \kappa_R d_R = r + q.$$

Proof. Checking on a basis and applying Cartan's formula, we have

$$\begin{aligned}(d_L\kappa_L + \kappa_L d_L)(f\,dx^{I,J}) &= \big((d\kappa + \kappa d)f\,dx^I\big) \otimes dx^J \\ &= \big(\mathfrak{L}_{X_{\mathrm{id}}}(f\,dx^I)\big) \otimes dx^J \\ &= \big((r+p)(f\,dx^I)\big) \otimes dx^J.\end{aligned}$$

In the last step, we used $\mathfrak{L}_{X_{\mathrm{id}}} x^i = x^i$ and hence $\mathfrak{L}_{X_{\mathrm{id}}} dx^i = dx^i$, so, using the Leibniz rule, the Lie derivative applied to a differential form with homogeneous polynomial coefficients simply multiplies the form by the total degree, that is, the sum of the polynomial degree and the form degree.

The claim for the operators on the right factor is analogous. □

Proposition 3.21. *If $\varphi \in \mathcal{H}_r\Lambda^{p,q}_m$ and $\kappa_L\varphi = \kappa_R\varphi = 0$, then*

$$\begin{aligned}\kappa_L\kappa_R d_L d_R\varphi &= ((r+p)(r+q-1) - m(m+p-q+1))\,\varphi \\ &= (r+p+m)(r+q-m-1)\varphi.\end{aligned}$$

Proof. The idea is to use the commutation relations to move the κ_L and κ_R operators to the right to get zero. We compute

$$\begin{aligned}\kappa_L\kappa_R d_L d_R\varphi &= \kappa_R\kappa_L d_R d_L\varphi \\ &= (\kappa_R d_R\kappa_L d_L - \kappa_R s^* d_L)\varphi \\ &= ((\kappa_R d_R(r+p) - \kappa_R d_R d_L\kappa_L) - (\kappa_L d_L - s^*\kappa_R d_L))\varphi \\ &= (((r+p)(r+q) - (r+p)d_R\kappa_R - 0) \\ &\qquad - (((r+p) - d_L\kappa_L) - (s^* d_L\kappa_R - s^* s)))\varphi \\ &= ((r+p)(r+q) - (r+p) - s^* s)\varphi \\ &= ((r+p)(r+q-1) - m(m+p-q+1))\varphi \\ &= (r+p+m)(r+q-m-1).\end{aligned}$$

□

Proposition 3.22. *Let φ be a nonzero element of $\mathcal{H}_r\Lambda^{p,q}_m$. We have that $\kappa_L\varphi = \kappa_R\varphi = 0$ if and only if exactly one of the following holds:*

- *$r = p = q = m = 0$, so φ is a constant scalar field.*
- *$r = 1$, $m = q$, and $\varphi = i^{p,q}\kappa\psi$ for some $\psi \in \mathcal{H}_{r-1}\Lambda^{k+1}$, where $k = p+q$ and $i^{p,q}$ is defined in Definition 2.31.*
- *$r \geq 2$, and $\varphi \in \mathcal{H}^-_r\Lambda^{p,q}_m$.*

In the last case, we have

$$\varphi = \kappa_L\kappa_R\left(C^{-1} d_L d_R\varphi\right),$$

where

$$C = (r+p+m)(r+q-m-1).$$

Proof. It is easy to check that, in any of these three cases, $\kappa_L\varphi = \kappa_R\varphi = 0$. In the first case, φ is a $(0,0)$-form, so $\kappa_L\varphi = \kappa_R\varphi = 0$. In the second case, it is easy to check from the definition of $i^{p,q}$ that $\kappa_L i^{p,q} = i^{p-1,q}\kappa$ and $\kappa_R i^{p,q} = (-1)^p i^{p,q-1}\kappa$, so $\kappa_L\varphi = i^{p-1,q}\kappa^2\psi = 0$ and $\kappa_R\varphi = (-1)^p i^{p,q-1}\kappa^2\psi = 0$. Finally, in the third case, by definition, $\varphi = \kappa_L\kappa_R\psi$ for some ψ, so φ is in the kernel of κ_L and κ_R because the two operators commute and square to zero.

Assume now that φ is in the kernel of both κ_L and κ_R. We must prove that we are in one of the three cases. If $r = 0$, then φ is constant, and so $d_L\varphi = d_R\varphi = 0$. Along with the assumption that $\kappa_L\varphi = \kappa_R\varphi = 0$, Proposition 3.20 tells us that $r+p = r+q = 0$, from which we conclude that $p = q = r = 0$, so $\mathcal{H}_r\Lambda^{p,q}$ is simply the space of constant scalar fields. We also have $m = 0$ since $0 \leq m \leq q$.

Assume henceforth that $r \geq 1$. Since $r \geq 1$, the factor $r + p + m$ of C must be positive. Recall that, because $\Lambda^{p,q}_m$ is nonempty, we have $m \leq q$. So, the second factor $r + q - m - 1$ is positive except when $r = 1$ and $m = q$. So, apart from the case $r = 1$ and $m = q$, we have $C > 0$.

If $C > 0$, then Proposition 3.21 tells us that

$$\varphi = \kappa_L\kappa_R\left(C^{-1}d_Ld_R\varphi\right).$$

Since d_L and d_R lower polynomial degree by one, we have that $C^{-1}d_Ld_R\varphi \in \mathcal{H}_{r-2}\Lambda^{p+1,q+1}$, so $\varphi \in \mathcal{H}^-_r\Lambda^{p,q}$, as desired. In particular, $r \geq 2$.

So then it remains to consider the case $r = 1$. In this case, $d_Ld_R\varphi = 0$ because d_Ld_R lowers polynomial degree by two, so $C = 0$ by Proposition 3.21. As discussed, $C = 0$ implies $r = 1$ and $m = q$. Since $m = q$, by Proposition 2.35, we have that $\varphi = i^{p,q}\varphi'$ for some k-form φ. Since φ and φ' are equal as k-tensors, φ' likewise has homogeneous polynomial coefficients of degree r. We claim that $\kappa\varphi' = 0$. This claim is trivial if $k = 0$. Otherwise, $p \geq 1$ or $q \geq 1$. If $p \geq 1$, then we use $0 = \kappa_L\varphi = i^{p-1,q}\kappa\varphi'$, which implies that $\kappa\varphi' = 0$ because $i^{p-1,q}$ is an inclusion. If $q \geq 1$, we reason similarly using κ_R. By Cartan's formula, we have $(d\kappa + \kappa d)\varphi' = (r + k)\varphi'$, so, using $r \geq 1$ and $\kappa\varphi' = 0$, we have $\varphi' = \kappa\psi$, where $\psi = (r + k)^{-1}\, d\varphi'$. Since d lowers polynomial degree, we have that $\psi \in \mathcal{H}_{r-1}\Lambda^{k+1}$, as desired. □

4. Extending double forms on the simplex

For it to be possible to construct finite element spaces of double forms, a key requirement is that we be able to extend a double form with vanishing trace on the standard simplex T^n to a double form with vanishing trace on $\mathbb{R}^{n+1}$. As we will see, doing so is possible except when $r = 0$ and $m = q$. We begin with definitions.

Definition 4.1. Let T^n denote the standard simplex in $\mathbb{R}^{n+1}$. Specifically,

$$T^n = \{(\lambda_0, \ldots, \lambda_n) \mid \lambda_i \geq 0, \sum_i \lambda_i = 1\}.$$

Definition 4.2. Let $\mathcal{P}_r(T^n)$ denote the space of polynomials on T^n of degree at most r. We define the spaces $\mathcal{P}_r\Lambda^k(T^n)$, $\mathcal{P}_r\Lambda^{p,q}(T^n)$, and $\mathcal{P}_r\Lambda^{p,q}_m(T^n)$ to be the corresponding spaces of forms or double forms with polynomial coefficients of degree at most r.

Definition 4.3. We have a natural inclusion of the boundary $\partial T^n \hookrightarrow T^n$. We say that a form or double form has *vanishing trace* if it vanishes when pulled back under this inclusion, or, equivalently, that the tensor vanishes at ∂T^n on vectors tangent to ∂T^n. We let $\mathring{\mathcal{P}}_r\Lambda^k(T^n)$, $\mathring{\mathcal{P}}_r\Lambda^{p,q}(T^n)$, and $\mathring{\mathcal{P}}_r\Lambda^{p,q}_m(T^n)$ to be the vanishing trace subspaces of the corresponding space.

Note that the boundary of ∂T^n is the set of points $(\lambda_0, \ldots, \lambda_n)$ in T^n such that $\lambda_i = 0$ for some i. This observation motivates the following definition.

Definition 4.4. For each i, we have a natural inclusion of the coordinate hyperplanes $\{\lambda_i = 0\} \hookrightarrow \mathbb{R}^{n+1}$. We say that a form or double form has *vanishing trace* if it vanishes when pulled back under this inclusion for all i. We let $\mathring{\mathcal{H}}_r\Lambda^k(\mathbb{R}^{n+1})$, $\mathring{\mathcal{H}}_r\Lambda^{p,q}(\mathbb{R}^{n+1})$, and $\mathring{\mathcal{H}}_r\Lambda^{p,q}_m(\mathbb{R}^{n+1})$ be the vanishing trace subspaces of the corresponding space.

Pulling back via the inclusion $T^n \hookrightarrow \mathbb{R}^{n+1}$, we can restrict a double form on $\mathbb{R}^{n+1}$ to a double form on T^n. Extension is the inverse of this operation.

Definition 4.5. We say that a double form φ on $\mathbb{R}^{n+1}$ is an *extension* of a double form $\bar{\varphi}$ on T^n if $\bar{\varphi}$ is the pull back of φ via the inclusion $T^n \hookrightarrow \mathbb{R}^{n+1}$.

Without the vanishing trace condition, extension is easy.

Proposition 4.6. *Every form in $\mathcal{P}_r\Lambda^{p,q}_m(T^n)$ can be extended to a form in $\mathcal{H}_r\Lambda^{p,q}_m(\mathbb{R}^{n+1})$.*

Proof. Observe that $\mathcal{H}_r\Lambda^{p,q}_m(\mathbb{R}^{n+1}) = \mathcal{H}_r(\mathbb{R}^{n+1}) \otimes \bigwedge^{p,q}_m V^*$, where $V = T_x\mathbb{R}^{n+1}$. Note that V is itself just $\mathbb{R}^{n+1}$, and hence independent of x, but we use the notation V to maintain the distinction between $\mathbb{R}^{n+1}$ as a vector space and $\mathbb{R}^{n+1}$ as a manifold. Likewise, $\mathcal{P}_r\Lambda^{p,q}_m(T^n) = \mathcal{P}_r(T^n) \otimes \bigwedge^{p,q}_m H^*$, where $H = T_xT^n$, a hyperplane of V. As a result, we can prove the proposition by proving two independent claims: The first claim is that polynomials on T^n can be extended to *homogeneous* polynomials on $\mathbb{R}^{n+1}$. The second claim is that the vector space map $\bigwedge^{p,q}_m V^* \to \bigwedge^{p,q}_m H^*$ is surjective.

The extension of polynomials is the standard homogenization procedure. Given a polynomial $\bar{f} \in \mathcal{P}_r(T^n)$, we can write it as a sum of monomials in the variables $\lambda_1, \dots, \lambda_n$ of degrees varying from 0 to r. We obtain $f \in \mathcal{H}_r(\mathbb{R}^{n+1})$ by multiplying each term by an appropriate power of $\lambda_0 + \cdots + \lambda_n$ so that the resulting term has degree exactly r. Since $\lambda_0 + \cdots + \lambda_n = 1$ on T^n, the polynomial f has the same values on T^n as $\bar{f}$.

For the linear algebra problem, since $H \hookrightarrow V$ is injective, we have that $V^* \to H^*$ is surjective, and hence so is $\bigwedge^{p,q} V^* \to \bigwedge^{p,q} H^*$. The compatibility with the decomposition follows from the fact that pull back respects the decomposition in Proposition 3.4, (technically, interpreting $\bigwedge^{p,q} V^*$ and $\bigwedge^{p,q} H^*$ as the space of constant double forms on $\mathbb{R}^{n+1}$ and T^n, respectively), and reasoning about surjective decomposition-respecting maps as in the proof of Proposition 3.19. □

With the vanishing trace condition, the question is more complicated. Certainly, double forms on $\mathbb{R}^{n+1}$ with vanishing trace restrict to double forms on T^n with vanishing trace. However, this map need not be surjective. As we will see, if $r = 0$ and $m = q$, then it is generally not be possible to extend a double form in $\mathring{\mathcal{P}}_r\Lambda^{p,q}_m(T^n)$ to a double form in $\mathring{\mathcal{H}}_r\Lambda^{p,q}_m(\mathbb{R}^{n+1})$. However, as we will also see, apart from this exceptional case, extension is always possible, via an explicit construction.

This construction relies on some ideas from [4]; we briefly review the key ideas we will need.

4.1. **The simplex, the sphere, and the Hodge star.** One of the key ideas from [4] is a coordinate transformation between the simplex and the sphere:

Definition 4.7. Let $\Phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be defined by

$$(\lambda_0, \dots, \lambda_n) = \Phi(u_0, \dots, u_n) = (u_0^2, \dots, u_n^2).$$

Noting that $\lambda_i \geq 0$ and that $u_0^2 + \cdots + u_n^2 = 1$ is equivalent to $\lambda_0 + \cdots + \lambda_n = 1$, we see that Φ maps the unit sphere S^n to the standard simplex T^n.

Notation 4.8. Because of the presence of squares, we will henceforth use subscript notation for coordinates, rather than the Einstein notation of superscripts and subscripts.

As we will see, one of the key benefits of this coordinate transformation is that it turns vanishing trace into full vanishing on the coordinate hyperplanes. To illustrate, observe that $d\lambda_i$ has vanishing trace on the hyperplane $\{\lambda_i = 0\}$. Indeed, $d\lambda_i$ vanishes on any vector tangent to the hyperplane. However, it does not vanish on vectors that are not tangent to the hyperplane, such as $\frac{\partial}{\partial \lambda_i}$. In contrast, the pull back of $d\lambda_i$ under the transformation $\lambda_i = u_i^2$ is $2u_i\, du_i$, which is identically zero on the hyperplane $\{u_i = 0\}$, vanishing on all vectors, not just those tangent to $\{u_i = 0\}$.

Another key idea from [4] is the relationship between the Hodge star on the sphere and the Koszul operator. To illustrate, observe that the Hodge star on one-forms on the two-sphere is just 90° rotation, which can be realized by taking the cross product with the normal vector. The normal vector on the sphere, however, is just the tautological vector field X_{id} in the definition of the Koszul operator.

Definition 4.9. We define the *tautological covector field*

$$\nu := \sum_{i=0}^{n} u_i\, du_i.$$

As the name suggests, $\nu = X_{\mathrm{id}}^\flat$ with respect to the standard metric on the $(u_0, \dots, u_n)$ coordinate system. As the notation suggests, restricted to the unit sphere, ν is the unit conormal.

Definition 4.10. We define an operator $\star_{S^n}\colon \Lambda^k(\mathbb{R}^{n+1}) \to \Lambda^{n-k}(\mathbb{R}^{n+1})$ by

$$\star_{S^n} \alpha := \star_{\mathbb{R}^{n+1}} (\nu \wedge \alpha),$$

where $\star_{\mathbb{R}^{n+1}}$ is the usual Hodge star operator on $\mathbb{R}^{n+1}$, with the subscript $\mathbb{R}^{n+1}$ added for clarity.

As the notation suggests, if we restrict to the sphere, then $\star_{S^n}$ is the Hodge star on the sphere.

Proposition 4.11 ([4, Proposition 2.16])**.** *Let α be a k-form on $\mathbb{R}^{n+1}$, and let $\bar{\alpha} \in \Lambda^k(S^n)$ be the pull back of α under the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$. Then*

$\star_{S^n} \bar{\alpha}$ is the pull back of $\star_{S^n} \alpha$, where $\star_{S^n} \bar{\alpha}$ refers to the Hodge star operator on the sphere, and $\star_{S^n} \alpha$ refers to Definition 4.10.

The Koszul operator κ is the contraction with X_{id}, which is adjoint to wedging with ν, yielding the following relationship.

Proposition 4.12. *For $\alpha \in \Lambda^k(\mathbb{R}^{n+1})$, we have*

$$\star_{S^n} \alpha = (-1)^k \kappa(\star_{\mathbb{R}^{n+1}} \alpha).$$

Proof. Since $\nu = X_{\mathrm{id}}^\flat$, we have $X_{\mathrm{id}} \lrcorner (\star_{\mathbb{R}^{n+1}} \alpha) = \star_{\mathbb{R}^{n+1}}(\alpha \wedge \nu)$; see, for example, [4, Proposition B.1]. We then compute

$$\kappa(\star_{\mathbb{R}^{n+1}} \alpha) = \star_{\mathbb{R}^{n+1}}(\alpha \wedge \nu) = (-1)^k \star_{\mathbb{R}^{n+1}}(\nu \wedge \alpha) = (-1)^k \star_{S^n} \alpha. \qquad \square$$

4.2. **An overview of the extension construction.** Before we proceed with the extension construction, we give an overview of how it will work, along with some examples.

Given a double form $\bar{\varphi}$ on T^n with vanishing trace, we can extend it to a double form φ with homogeneous coefficients on $\mathbb{R}^{n+1}$. Note that, by homogeneity, the fact that φ vanishes when pulled back to ∂T implies that it also vanishes when pulled back to any dilation $c\partial T$, where $c \in \mathbb{R}$. The union of these dilations is the union of the hyperplanes $\{\lambda_i = 0\}$, so one might ask why φ does not automatically vanish when pulled back to the hyperplanes $\{\lambda_i = 0\}$, which is the vanishing trace condition for $\mathbb{R}^{n+1}$. The answer is that the vanishing trace condition on $\mathbb{R}^{n+1}$ requires that φ vanish at the hyperplane for all vectors tangent to the hyperplane. On the other hand, we only have vanishing on vectors tangent to the dilates $c\partial T$, so φ does not have to vanish if we input a vector that is tangent to $\{\lambda_i = 0\}$ but not tangent to $c\partial T$. As we will see, this issue is the key issue that needs to be resolved to construct an extension with vanishing trace.

So, then, we proceed by computing the pull back $\psi = \Phi^*\varphi$. Letting $\bar{\psi} = \Phi^*\bar{\varphi}$ we have that $\bar{\psi}$ is the pull back of ψ to S^n via $S^n \hookrightarrow \mathbb{R}^{n+1}$. As we discussed, $\bar{\psi}$ has vanishing trace in a stronger sense. Specifically, at $S^n \cap \{u_i = 0\}$, $\bar{\psi}$ vanishes on all vectors tangent to S^n, not just those tangent to $S^n \cap \{u_i = 0\}$. By homogeneity, we conclude that ψ vanishes at $\{u_i = 0\}$ on all vectors tangent to the dilates cS^n. However, as before, in general, ψ will not vanish if we input a vector that is not tangent to cS^n, such as, for example, X_{id}.

So, now we apply $\circledast_{S^n}$, defined by applying $\star_{S^n}$ to both factors of the double form. At points in $S^n \cap \{u_i = 0\}$, since ψ vanishes on all vectors tangent to S^n, so does $\circledast_{S^n} \psi$. However, by Proposition 4.12, we see that $\circledast_{S^n} \psi = (-1)^k \kappa_L \kappa_R \circledast_{\mathbb{R}^{n+1}} \psi$, where $k = p + q$. Therefore, $\circledast_{S^n} \psi$ is in the image of both κ_L and κ_R, and hence in the kernel of both κ_L and κ_R, which we recall are contraction with X_{id}. Thus, unlike ψ, we have that $\circledast_{S^n} \psi$ vanishes if we input X_{id}. Since X_{id} along with the vectors tangent to S^n span the entire tangent space to $\mathbb{R}^{n+1}$, we conclude that $\circledast_{S^n} \psi$ is zero on all vectors at points in $S^n \cap \{u_i = 0\}$. By homogeneity, we conclude that $\circledast_{S^n} \psi$

is zero on all vectors at all points in the hyperplane $\{u_i = 0\}$. Consequently, all of the polynomial coefficients of $\circledast_{S^n} \psi$ are divisible by u_i, so $\circledast_{S^n} \psi$ is divisible by $u_N := u_0 \cdots u_n$.

So now we divide by u_N and consider $u_N^{-1} \circledast_{S^n} \psi$. By this point, our double form on S^n is very different from what we started with, so our task now is to undo this whole process as far as S^n is concerned. Since $\circledast_{S^n} = (-1)^k \kappa_L \kappa_R \circledast_{\mathbb{R}^{n+1}}$ and $\circledast_{\mathbb{R}^{n+1}}$ is invertible, the task amounts to inverting $\kappa_L \kappa_R$. Proposition 3.22 is exactly the tool for the job. Since $\circledast_{S^n} \psi$ is in the kernel of both κ_L and κ_R, so is $u_N^{-1} \circledast_{S^n} \psi$, so Proposition 3.22 applies, and we have $u_N^{-1} \circledast_{S^n} \psi = \kappa_L \kappa_R C^{-1} d_L d_R (u_N^{-1} \circledast_{S^n} \varphi)$, so $(-1)^k \circledast_{R^{n+1}}^{-1} C^{-1} d_L d_R \varphi$ is the desired inverse image of $u_N^{-1} \circledast_{S^n} \psi$ under $\circledast_{S^n}$. Note that $\circledast_{S^n}$ is not injective on double forms on $\mathbb{R}^{n+1}$, so we do not simply get $u_N^{-1}\psi$. On the other hand, $\circledast_{S^n}$ is certainly bijective on double forms on S^n, so the restriction to S^n is indeed simply $u_N^{-1}\bar{\psi}$.

Our penultimate step is simply to multiply back by u_N. Then, restricted to the sphere, we have $\bar{\psi}$. Meanwhile, on $\mathbb{R}^{n+1}$, we have something that, being a multiple of u_N, manifestly vanishes on the hyperplanes. Pushing forward via Φ, we obtain an extension of $\bar{\varphi}$ that has vanishing trace on the hyperplanes, as desired.

In the remainder of this section, we will prove that each step works as described in this overview, but we first provide an example and a counterexample.

Example 4.13. Let $n = 1$, and let ds be the length element of T^1, normalized so that the length of T^1 is one. Let $\bar{\varphi} = ds \otimes ds$. Since we have a $(1,1)$-form and the boundary of T^1 is zero-dimensional, we know that $\bar{\varphi}$ has vanishing trace. Since $\bar{\varphi}$ is symmetric, we have $m = 0$. Our goal is to construct an extension of $\bar{\varphi}$ to $\mathbb{R}^2$ that has vanishing trace to the hyperplanes $\{\lambda_0 = 0\}$ and $\{\lambda_1 = 0\}$.

(1) We first construct an arbitrary extension of $\bar{\varphi}$ to $\mathcal{H}_0\Lambda_0^{1,1}(\mathbb{R}^2)$. In this case, $\varphi = d\lambda_1 \otimes d\lambda_1$ suffices. Note that, while φ vanishes on $\{\lambda_1 = 0\}$, it does *not* vanish on $\{\lambda_0 = 0\}$. Our goal is to find an extension that does.
(2) We pull back via Φ. Since $d\lambda_1 = 2u_1\, du_1$, we obtain $4u_1^2\, du_1 \otimes du_1$.
(3) We apply $\circledast_{S^n}$.
   (a) Applying $(\nu \otimes \nu)\owedge$, we obtain
   $$4u_0^2 u_1^2 (du_0 \wedge du_1) \otimes (du_0 \wedge du_1).$$
   (b) Applying $\circledast_{\mathbb{R}^{n+1}}$, we obtain $4u_0^2u_1^2$.
   Note that we could also compute using $\circledast_{S^n} = (-1)^k \kappa_L \kappa_R \circledast_{\mathbb{R}^{n+1}}$.
(4) We divide by $u_N = u_0 u_1$, obtaining $4u_0u_1$.
(5) We divide by C. In the formula for C, we have $r = 2$ and $p = q = m = 0$, so $C = 2$, so we obtain $2u_0u_1$.
(6) We apply $d_L d_R$, obtaining $2(du_0 \otimes du_1 + du_1 \otimes du_0)$.
(7) We apply $(-1)^k \circledast_{R^{n+1}}^{-1}$. We obtain $-2(du_1 \otimes du_0 + du_0 \otimes du_1)$.

(8) We multiply by u_N, obtaining
$$-2u_0u_1(du_1 \otimes du_0 + du_0 \otimes du_1).$$
(9) We push forward via Φ, obtaining
$$-\tfrac{1}{2}(d\lambda_1 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_1).$$

It is clear that the result vanishes if we pull back to $\{\lambda_0 = 0\}$ so $d\lambda_0 = 0$, and likewise if we pull back to $\{\lambda_1 = 0\}$ so $d\lambda_1 = 0$. Note that the result exactly matches the Regge basis $-\frac{1}{2}\,d\lambda_i \odot d\lambda_j$.

Example 4.14. Let $n = 2$, and consider the area form on T^2, normalized so that T^2 has area one, interpreted as an antisymmetric $(1, 1)$-form. This tensor has vanishing trace, but the construction *fails* because we are in the exceptional case $r = 0$ and $m = q$; no extension exists. It is illustrative to see what goes wrong.

(1) We begin with an arbitrary extension; $2(d\lambda_1 \otimes d\lambda_2 - d\lambda_2 \otimes d\lambda_1)$ suffices.
(2) We pull back via Φ, obtaining
$$8u_1u_2(du_1 \otimes du_2 - du_2 \otimes du_1).$$
(3) We apply $\circledast_{S^n}$.
   (a) Applying $(\nu \otimes \nu)\owedge$, we obtain
$$\begin{aligned}8u_1u_2(&(u_0\,du_0 \wedge du_1 + u_2\,du_2 \wedge du_1) \otimes (u_0\,du_0 \wedge du_2 + u_1\,du_1 \wedge du_2)\\ &- (u_0\,du_0 \wedge du_2 + u_1\,du_1 \wedge du_2) \otimes (u_0\,du_0 \wedge du_1 + u_2\,du_2 \wedge du_1)).\end{aligned}$$
   (b) Applying $\circledast_{\mathbb{R}^{n+1}}$, we obtain
$$\begin{aligned}8u_1u_2(&(u_0\,du_2 - u_2\,du_0) \otimes (-u_0\,du_1 + u_1\,du_0)\\ &- (-u_0\,du_1 + u_1\,du_0) \otimes (u_0\,du_2 - u_2\,du_0)),\end{aligned}$$
   which, with cancellation, simplifies to
$$\begin{aligned}8u_0u_1u_2(&u_0(du_1 \otimes du_2 - du_2 \otimes du_1)\\ &+ u_1(du_2 \otimes du_0 - du_0 \otimes du_2)\\ &+ u_2(du_0 \otimes du_1 - du_1 \otimes du_0)).\end{aligned}$$
(4) Dividing by u_N yields
$$\begin{aligned}8(&u_0(du_1 \otimes du_2 - du_2 \otimes du_1)\\ &+ u_1(du_2 \otimes du_0 - du_0 \otimes du_2)\\ &+ u_2(du_0 \otimes du_1 - du_1 \otimes du_0)).\end{aligned}$$
(5) In the formula for C, we have $r = p = q = m = 1$, so $C = 0$, so we cannot divide by C. Indeed, we are in the exceptional case of Proposition 3.22, where we are in the kernel of κ_L and κ_R but fail to be in the image of $\kappa_L\kappa_R$. Alternatively, we can see that we will fail because d_Ld_R will yield zero because our expression has polynomial degree one and d_Ld_R lowers polynomial degree by two.

We now proceed with discussing each of the operations in the construction in detail.

4.3. **The pull back, vanishing trace, and even double forms.** We begin by investigating the pull back operation Φ^* given by $\lambda_i = u_i^2$, $d\lambda_i = 2u_i\, du_i$.

Proposition 4.15. *The pull back Φ^* is an injective map from $\Lambda^{p,q}(T^n)$ to $\Lambda^{p,q}(S^n)$.*

Proof. Observe that Φ is a diffeomorphism from the part of S^n in the positive orthant to the interior of T^n. So, therefore, if $\bar{\psi} \in \Lambda^{p,q}(T^n)$ and $\Phi^*\bar{\psi} = 0$, then $\bar{\psi}$ is zero on the interior of T^n. Since $\bar{\psi}$ is smooth, it must therefore be zero on the boundary of T^n, too. □

Proposition 4.16. *The pull back Φ^* is an injective map from $\mathcal{H}_r\Lambda_m^{p,q}(\mathbb{R}^{n+1})$, to $\mathcal{H}_{2r+k}\Lambda_m^{p,q}(\mathbb{R}^{n+1})$, where $k = p + q$.*

Proof. Let $\varphi \in \mathcal{H}_r\Lambda_m^{p,q}(\mathbb{R}^{n+1})$, and $\psi = \Phi^*\varphi$. Because $\lambda_i = u_i^2$, the pull back ψ gets two polynomial degrees per polynomial degree of φ; additionally, from $d\lambda_i = 2u_i\, du_i$, ψ acquires one polynomial degree for every form degree of φ. Pull back respects the decomposition by Proposition 3.4.

The proof of injectivity is similar to above. Observe that Φ is a diffeomorphism if we restrict the domain and codomain to the strictly positive orthant of $\mathbb{R}^{n+1}$. Therefore, if $\psi = 0$, we can conclude that $\varphi = 0$ on the strictly positive orthant. Since φ has polynomial coefficients, the fact that φ vanishes on an open set implies that it vanishes on all of $\mathbb{R}^{n+1}$. □

Our construction also requires that we invert the pull back operation, but doing so is not always possible, even for scalar fields. For example, u_0u_1 gets pushed forward to $\sqrt{\lambda_0\lambda_1}$, which is not a polynomial. As such, we need additional conditions.

Definition 4.17. Let R_i be the reflection across the coordinate plane $\{u_i = 0\}$, so $u_i \mapsto -u_i$. We say that a double form ψ is *even* if $R_i^*\psi = \psi$ for all i.

Since $u_i \mapsto -u_i$ yields $du_i \mapsto -\,du_i$, to check if a polynomial double form is even, in each term, for each i, we count the total number of times u_i or du_i appears; this total must be even.

Proposition 4.18. *If $\varphi \in \Lambda^{p,q}(\mathbb{R}^{n+1})$ and $\psi = \Phi^*\varphi \in \Lambda^{p,q}(\mathbb{R}^{n+1})$, then ψ is even.*

Proof. Since $\lambda_i = u_i^2 = (-u_i)^2$, we have that $\Phi \circ R_i = \Phi$, so $R_i^*\Phi^*\varphi = \Phi^*\varphi$, so $R_i^*\psi = \psi$. □

We might guess that perhaps we find a preimage of Φ^* for *even* double forms. However, unlike the case of simple forms in [4], even the even condition is not enough. For example $u_1^2\, du_0 \otimes du_0$ is even, but it gets pushed forward to $\frac{\lambda_1}{4\lambda_0}\, d\lambda_0 \otimes d\lambda_0$, which is not a polynomial. However, for our construction, we will only need to push forward double forms that are not only even but also divisible by $u_N := u_0 \cdots u_n$. We will see that not only is the push forward a polynomial double form, it also has vanishing trace.

Notation 4.19. Let $u_N := \prod_{i=0}^n u_i$ denote the product of the coordinate functions.

Proposition 4.20. *Assume that $\psi \in \mathcal{H}_{2r+k}\Lambda^{p,q}_m(\mathbb{R}^{n+1})$ is even and divisible by u_N. Then $\psi = \Phi^*\varphi$ for a unique $\varphi \in \mathring{\mathcal{H}}_r\Lambda^{p,q}_m(\mathbb{R}^{n+1})$ with vanishing trace.*

Proof. Let $\psi = u_N\psi'$. Then ψ' is odd in the sense that $R_i^*\psi' = -\psi'$. Consider a term $f\,du_{I,J}$ of ψ', where f is a monomial. Consider the case where this term contains zero or two copies of du_i, that is, i is in both I and J or in neither of them. Then, for ψ' to be odd, this term must also contain an odd power of u_i in the polynomial factor f; in particular, this power must be at least one. Consequently, the corresponding term $u_N f\,du_{I,J}$ in φ has an even power of u_i that is at least two. As a result, if we had two copies of du_i, we can match up each du_i with a u_i, and we can push forward each $u_i\,du_i$ to $\frac{1}{2}\,d\lambda_i$, leaving behind an even power of u_i, which pushes forward to an integer power of λ_i. If we had zero copies of du_i, then we just have a positive even power of u_i, which pushes forward to a positive integer power of λ_i. In particular, the push forward must have at least one λ_i or $d\lambda_i$.

Meanwhile, a term $u_N f\,du_{I,J}$ of ψ that contains one copy of du_i must also have an odd power of u_i in the polynomial factor $u_N f$ because ψ is even. We likewise have that $u_i\,du_i$ pushes forward to $\frac{1}{2}\,d\lambda_i$, leaving behind an even power of u_i, which pushes forward to an integer power of λ_i.

Note that, in either case, for every i, the term of the push forward has at least one λ_i or $d\lambda_i$, so it vanishes when pulled back to the hyperplane $\{\lambda_i = 0\}$, as required by the definition of vanishing trace on $\mathbb{R}^{n+1}$.

Thus, there exists a push forward $\varphi \in \mathring{\mathcal{H}}_r\Lambda^{p,q}(\mathbb{R}^{n+1})$. The push forward is unique by because Φ^* is injective. With regards to the decomposition, to check that $\varphi \in \Lambda^{p,q}_m$, we can let φ_m be the projection of $\varphi \in \Lambda^{p,q}$ onto the $\Lambda^{p,q}_m$ summand. By Proposition 3.4, since $\psi \in \Lambda^{p,q}_m$, we have that $\Phi^*\varphi_m = \psi$. By the uniqueness of φ, we have $\varphi = \varphi_m$. □

We now prove that vanishing trace on T^n (vanishing of tangential components on ∂T^n) yields full vanishing (all components) on the great circles of S^n.

Proposition 4.21. *Let $\bar{\varphi} \in \Lambda^{p,q}(T^n)$ and let $\bar{\psi} = \Phi^*\bar{\varphi} \in \Lambda^{p,q}(S^n)$. If $\bar{\varphi}$ has vanishing trace, then $\bar{\psi}$ fully vanishes on the great circles $\{u_i = 0\}$, in the sense that it vanishes on all vectors tangent to S^n, not just those vectors tangent to the great circle.*

Proof. Let $u = (u_0, \dots, u_n)$ be a point on S^n, and assume that $u_i = 0$. Let $\lambda = \Phi(u)$, a point on the boundary component $\{\lambda_i = 0\}$ of T^n.

Let e_i denote the ith coordinate basis vector at u, that is, $e_i = \frac{\partial}{\partial u_i}\big|_u$. Because $u_i = 0$, we have $e_i \in T_uS^n$. Note that e_i is normal to the great circle $\{u_i = 0\}$, so any vector in T_uS^n can be written in the form $be_i + X$, where b is a real number and X is tangent to the great circle $\{u_i = 0\}$.

Observe that the push forward $\Phi_* e_i$ is zero. Indeed, $\frac{\partial}{\partial u_i} = \frac{\partial \lambda_i}{\partial u_i}\frac{\partial}{\partial \lambda_i} = 2u_i\frac{\partial}{\partial \lambda_i}$, which is zero at u. Meanwhile, vectors tangent to the great circle $\{u_i = 0\}$ get pushed forward to vectors tangent to the boundary component $\{\lambda_i = 0\}$ of T^n. So, at u, applying $\bar{\psi}$ to vectors in T_uS^n written in the form $be_i + X$, we obtain

$$\begin{aligned}&\bar{\psi}|_u(b_1e_i + X_1, \ldots, b_pe_i + X_p; c_1e_i + Y_1, \ldots, c_qe_i + Y_q)\\&= \bar{\varphi}|_\lambda(\Phi_*(b_1e_i + X_1), \ldots, \Phi_*(b_pe_i + X_p); \Phi_*(c_1e_i + Y_1), \ldots, \Phi_*(c_qe_i + Y_q))\\&= \bar{\varphi}|_\lambda(\Phi_*X_1, \ldots, \Phi_*X_p; \Phi_*Y_1, \ldots, \Phi_*Y_q),\end{aligned}$$

which is zero because $\bar{\varphi}$ has vanishing trace and the Φ_*X_a and Φ_*Y_a are tangent to the boundary component $\{\lambda_i = 0\}$. □

4.4. **The double Hodge star on the sphere.** Recall from Definition 3.2 that we have double Hodge star operations $\circledast_{S^n} \colon \Lambda^{p,q}(S^n) \to \Lambda^{n-p,n-q}(S^n)$ and $\circledast_{\mathbb{R}^{n+1}} \colon \Lambda^{p,q}(\mathbb{R}^{n+1}) \to \Lambda^{n+1-p,n+1-q}(\mathbb{R}^{n+1})$ by applying $\star$ to each factor of the double form. Recall from Definition 4.10 that we defined $\star_{S^n}$ on differential forms on $\mathbb{R}^{n+1}$, so we can analogously define $\circledast_{S^n}$ on double forms on $\mathbb{R}^{n+1}$ as well.

Definition 4.22. Define $\circledast_{S^n} \colon \Lambda^{p,q}(\mathbb{R}^{n+1}) \to \Lambda^{n-p,n-q}(\mathbb{R}^{n+1})$ on simple tensors by

$$\circledast_{S^n}(\alpha \otimes \beta) := (\star_{S^n} \alpha) \otimes (\star_{S^n} \beta).$$

Each proposition in Section 4.1 yields analogous propositions for double forms.

Proposition 4.23. *Let ψ be a (p, q)-form on $\mathbb{R}^{n+1}$ and let $\bar{\psi}$ be its pull back under the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$. Then $\circledast_{S^n} \bar{\psi}$ is the pull back of $\circledast_{S^n} \psi$, where $\circledast_{S^n} \bar{\psi}$ refers to Definition 3.2 and $\circledast_{S^n} \varphi$ refers to Definition 4.22.*

Proof. On simple tensors, the claim follows by applying Proposition 4.11 to each factor, and then we extend by linearity. □

Proposition 4.24. *For $\psi \in \Lambda^{p,q}(\mathbb{R}^{n+1})$, we have*

$$\circledast_{S^n} \psi = (-1)^k \kappa_L \kappa_R (\circledast_{\mathbb{R}^{n+1}} \varphi),$$

where $k = p + q$.

Proof. As before, on simple tensors, the claim follows by applying Proposition 4.12 to each factor. □

On the sphere, $\circledast_{S^n}$ preserves fully vanishing on great circles. However, on Euclidean space, it yields full vanishing on hyperplanes.

Proposition 4.25. *Let $\psi \in \mathcal{H}_r\Lambda^{p,q}(\mathbb{R}^{n+1})$, and let $\bar{\psi}$ be the pull back of ψ under the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$. Assume that $\bar{\psi}$ fully vanishes on the great circles of S^n, that is, at every point $u \in S^n$ with $u_i = 0$, we have that*

$\psi|_u(X_1,\dots,X_p;Y_1,\dots,Y_q)=0$ for all vectors X_a, Y_a in T_uS^n. Then $\circledast_{S^n}\psi$ is divisible by $u_N=\prod_{i=0}^n u_i$.

Proof. Let $u\in S^n$ with $u_i=0$. Then the double multicovector $\bar\psi|_u$ is identically zero on T_uS^n, and so $\circledast_{S^n}\bar\psi|_u$ is also identically zero on T_uS^n. Because $\circledast_{S^n}\bar\psi$ is the pull back of $\circledast_{S^n}\psi$, we conclude that $\circledast_{S^n}\psi|_u$ is identically zero on vectors in T_uS^n. We claim that it is in fact identically zero on all vectors in $T_u\mathbb{R}^{n+1}$.

Note that the tautological vector field $X_{\mathrm{id}}=\sum_{i=0}^n u_i\frac{\partial}{\partial u_i}$ is normal to the sphere, so any vector in $X\in T_u\mathbb{R}^{n+1}$ can be written as $X=bX_{\mathrm{id}}+\bar X$, where b is a real number and $\bar X$ is tangent to the sphere. Since $\circledast_{S^n}\psi$ is in the image of $\kappa_L\kappa_R$, it is in the kernel of both κ_L and κ_R. Consequently, by antisymmetry, the expression $\circledast_{S^n}\psi|_u(X_1,\dots,X_p;Y_1,\dots,Y_q)$ vanishes if any of the X_a or Y_a are the tautological vector field X_{id}. By multilinearity, writing each X_a and Y_a in the above form, we obtain

$$\begin{aligned}&\circledast_{S^n}\psi|_u(X_1,\dots,X_p;Y_1,\dots,Y_q)\\&=\circledast_{S^n}\psi|_u(b_1X_{\mathrm{id}}+\bar X_1,\dots,b_pX_{\mathrm{id}}+\bar X_p;c_1X_{\mathrm{id}}+\bar Y_1,\dots,c_qX_{\mathrm{id}}+\bar Y_q)\\&=\circledast_{S^n}\psi|_u(\bar X_1,\dots,\bar X_p;\bar Y_1,\dots,\bar Y_q),\end{aligned}$$

which is zero because $\circledast_{S^n}\psi|_u$ vanishes on vectors tangent to the sphere.

We have shown that $\circledast_{S^n}\psi|_u$ is the zero double multicovector at any point u on the sphere with $u_i=0$. In other words, in the standard form, all of the polynomial coefficients of $\circledast_{S^n}\psi$ vanish at this point u. Because the polynomial coefficients are homogeneous, they must also vanish at any scalar multiple of this point. Hence, the polynomial coefficients vanish on the entire plane $u_i=0$. Consequently, the polynomial coefficients must be divisible by u_i. □

4.5. **The extension construction.** We now have the tools to follow the steps outlined in Section 4.2 to construct extensions of double forms on T^n with vanishing trace, and to understand when the construction fails and the extension does not exist.

Theorem 4.26. *Let $\bar\varphi\in\mathring{\mathcal{P}}_r\Lambda^{p,q}_m(T^n)$ be nonzero. Let $\varphi\in\mathcal{H}_r\Lambda^{p,q}_m(\mathbb{R}^{n+1})$ be an arbitrary extension of $\bar\varphi$ to $\mathbb{R}^{n+1}$. Provided we are not in the case $r=0$ and $m=q$, then $\bar\varphi$ also has an extension $\varphi'\in\mathring{\mathcal{H}}_r\Lambda^{p,q}_m(\mathbb{R}^{n+1})$ with vanishing trace, given by the formula*

$$\varphi'=(-1)^kC^{-1}(\Phi^*)^{-1}u_N\circledast^{-1}_{\mathbb{R}^{n+1}}d_Ld_R(u_N^{-1}\circledast_{S^n}\Phi^*\varphi),$$

where $k=p+q$, $C=(2r+p+m+1)(2r+q-m)$, and $u_N=\prod u_i$.

Proof. We have all the ingredients, so now we just apply each operator step by step. Let $\psi=\Phi^*\varphi$ and $\bar\psi$ be its restriction to the sphere S^n, so we also have $\bar\psi=\Phi^*\bar\varphi$. By Proposition 4.16, we have $\psi\in\mathcal{H}_{2r+k}\Lambda^{p,q}_m(\mathbb{R}^{n+1})$ and $\bar\psi$ fully vanishes on the great circles $\{u_i=0\}$ by Proposition 4.21.

Next, we have that $\circledast_{S^n}\psi=(-1)^k\kappa_L\kappa_R\circledast_{\mathbb{R}^{n+1}}\psi$ by Proposition 4.24. Noting that $\circledast_{\mathbb{R}^{n+1}}$ does not change polynomial degree, by Proposition 2.49, we have that $\circledast_{\mathbb{R}^{n+1}} \psi \in \mathcal{H}_{2r+k}\Lambda^{n+1-p,n+1-q}_{m^*}(\mathbb{R}^{n+1})$, where $m^* = m+p-q$. Then, noting that κ_L and κ_R raise polynomial degree and lower form degree, we have by Proposition 3.12 that $\circledast_{S^n} \psi \in \mathcal{H}_{2r+k+2}\Lambda^{n-p,n-q}_{m^*}(\mathbb{R}^{n+1})$. Its restriction to the sphere is $\circledast_{S^n} \bar{\psi}$ by Proposition 4.23.

By Proposition 4.25, $\circledast_{S^n} \psi$ is divisible by $u_N = \prod_{i=0}^n u_i$. So, $u_N^{-1} \circledast_{S^n} \psi \in \mathcal{H}_{2r+k-n+1}\Lambda^{n-p,n-q}_{m^*}(\mathbb{R}^{n+1})$. Recalling that $\circledast_{S^n} \psi = (-1)^k \kappa_L \kappa_R \circledast_{\mathbb{R}^{n+1}} \psi$, we have that $\circledast_{S^n} \psi$ is in the kernel of both κ_L and κ_R. Since multiplication by u_N^{-1} commutes with κ_L and κ_R, we conclude that $u_N^{-1} \circledast_{S^n} \psi$ is in the kernel of κ_L and κ_R, too. So now we would like to apply Proposition 3.22 to show that $u_N^{-1} \circledast_{S^n} \psi$ is in the image of $\kappa_L\kappa_R$.

To do so, we must first deal with the exceptional cases of Proposition 3.22. First, Proposition 3.22 requires that $u_N^{-1} \circledast_{S^n} \psi$ be nonzero. Assume for the sake of contradiction that $u_N^{-1} \circledast_{S^n} \psi = 0$, so then $\circledast_{S^n} \psi$ would be zero. Although $\circledast_{S^n}$ is not injective on forms on $\mathbb{R}^{n+1}$, it is bijective on forms on S^n, so then we could conclude that $\bar{\psi}$ is zero, from which it would follow by Proposition 4.15 that $\bar{\varphi}$ is zero, which we assumed is not the case.

Since $u_N^{-1} \circledast_{S^n} \psi \in \mathcal{H}_{2r+k-n+1}\Lambda^{n-p,n-q}_{m^*}(\mathbb{R}^{n+1})$, the first case of Proposition 3.22 reads $2r+k-n+1 = n-p = n-q = m^* = 0$. In particular $p = n$, $q = n$, $k = p+q = 2n$, and so $2r+k-n+1 = 2r+n+1$, which cannot be zero.

The second case of Proposition 3.22 reads $2r+k-n+1 = 1$ and $m^* = n-q$. The first equation gives $n = 2r+k$, which implies $n \geq k$. Recalling that $m^* = m+p-q$, the second equation gives $n = m+p$. Recalling that $m \leq q$, we have $n \leq q+p = k$. We conclude that $n = k$, so $r = 0$, and $m = q$, which is the exceptional case excluded in the theorem statement. As we saw in Example 4.14 and will see more generally below, vanishing trace extension is not possible in this case.

So, we are in the general case of Proposition 3.22, so

$$u_N^{-1} \circledast_{S^n} \psi = C^{-1} \kappa_L \kappa_R d_L d_R (u_N^{-1} \circledast_{S^n} \psi), \tag{3}$$

where C is

$$\big((2r+k-n+1)+(n-p)+(m+p-q)\big)\big((2r+k-n+1)+(n-q)-(m+p-q)-1\big) = (2r+p+m+1)(2r+q-m).$$

So now let

$$\psi' = (-1)^k C^{-1} u_N \circledast^{-1}_{R^{n+1}} d_L d_R (u_N^{-1} \circledast_{S^n} \psi). \tag{4}$$

Per Proposition 3.15, $d_L d_R(u_N^{-1} \circledast_{S^n} \psi) \in \mathcal{H}_{2r+k-n-1}\Lambda^{n-p+1,n-q+1}_{m^*}$, so then $\circledast^{-1}_{\mathbb{R}^{n+1}} d_L d_R (u_N^{-1} \circledast_{S^n} \psi) \in \mathcal{H}_{2r+k-n-1}\Lambda^{p,q}_m$, and so $\psi' \in \mathcal{H}_{2r+k}\Lambda^{p,q}_m$. So then, applying Proposition 4.24, using the fact that multiplication by u_N commutes with pointwise operations κ_L, κ_R, and $\circledast_{\mathbb{R}^{n+1}}$, and using Equation (3), we obtain

$$\circledast_{S^n} \psi' = (-1)^k \kappa_L \kappa_R \circledast_{\mathbb{R}^{n+1}} \psi' = C^{-1} u_N \kappa_L \kappa_R d_L d_R (u_N^{-1} \circledast_{S^n} \psi) = \circledast_{S^n} \psi$$

Letting $\bar{\psi}'$ be the restriction of ψ' to the sphere, we conclude by Proposition 4.23 that $\circledast_{S^n} \bar{\psi}' = \circledast_{S^n} \bar{\psi}$, so $\bar{\psi}' = \bar{\psi}$ because $\circledast_{S^n}$ is bijective on the sphere.

The final step is to let $\varphi' = (\Phi^*)^{-1}\psi'$, but to do so we must verify that the push forward exists using Proposition 4.20. We know that ψ is even by Proposition 4.18. We recall the notation that R_i is the reflection across the hyperplane $\{u_i = 0\}$ given by $u_i \mapsto -u_i$. The Hodge star $\star_{\mathbb{R}^{n+1}}$ anticommutes with pull back under reflections, so then the double Hodge star $\circledast_{\mathbb{R}^{n+1}}$ commutes with pull back under reflections. The operations d_L and d_R commute with any pull back. Since the vector field X_{id} is invariant under reflection, κ_L and κ_R commute with R_i^*. Finally, $R_i^* u_N = -u_N$, so multiplication or division by u_N anticommutes with R_i^*. So, all of the operations in Equation (4) commute with R_i^*, with the exception of u_N and u_N^{-1}, each of which anticommutes with R_i^*. We conclude that ψ' is even. We have that ψ' is divisible by u_N by construction. So, by Proposition 4.20, there exists a unique $\varphi' \in \mathring{\mathcal{H}}_r\Lambda_m^{p,q}(\mathbb{R}^{n+1})$ with $\Phi^*\varphi' = \psi'$.

Letting $\bar{\varphi}'$ be the pull back of φ' to T^n, we have $\Phi^*\bar{\varphi}' = \bar{\psi}' = \bar{\psi} = \Phi^*\bar{\varphi}$, so $\bar{\varphi}' = \bar{\varphi}$ by Proposition 4.15. We conclude that φ' is the desired vanishing trace extension of $\bar{\varphi}$. ☐

We also show that extension fails in the exceptional case $r = 0$ and $m = q$.

Proposition 4.27. *Let $\bar{\varphi} \in \mathring{\mathcal{P}}_0\Lambda_q^{p,q}(T^n)$ be nonzero. Then there does not exist a vanishing trace extension $\varphi' \in \mathring{\mathcal{H}}_0\Lambda_q^{p,q}(\mathbb{R}^{n+1})$ of $\bar{\varphi}$.*

Proof. The initial part of the proof of Theorem 4.26 proceeds as before, with φ an arbitrary extension of $\bar{\varphi}$, then setting $\psi := \Phi^*\varphi$, and then finding that $u_N^{-1} \circledast_{S^n} \psi$ is a nonzero element of $\mathcal{H}_{2r+k-n+1}\Lambda_{m^*}^{n-p,n-q}$ that is in the kernel of κ_L and κ_R. The first case of Proposition 3.22 likewise yields a contradiction.

So, we are in the second or third case of Proposition 3.22. Plugging in $r = 0$ and $m = q$, we find that $u_N^{-1} \circledast_{S^n} \psi \in \mathcal{H}_{k-n+1}\Lambda_p^{n-p,n-q}$. Recalling that $\Lambda_m^{p,q}$ being nonzero implies $m \le q$, we have that $\Lambda_p^{n-p,n-q}$ being nonzero implies $p \le n-q$, so $k = p+q \le n$, and so the polynomial degree $k-n+1$ is at most 1. We conclude that we cannot be in the third case of Proposition 3.22, so we must be in the second case, and so $k - n + 1 = 1$ and $p = n - q$, both of which tell us that $k = n$.

We claim that, when $k = n$, the space $\mathring{\mathcal{H}}_0\Lambda_q^{p,q}(\mathbb{R}^{n+1})$ is zero and hence cannot contain an extension of a nonzero double form on T^n. By Proposition 2.35, the space $\mathring{\mathcal{H}}_0\Lambda_q^{p,q}(\mathbb{R}^{n+1})$ is the image of $\mathring{\mathcal{H}}_0\Lambda^k(\mathbb{R}^{n+1})$ under the inclusion $i^{p,q}$ of k-forms into (p,q)-forms, so it suffices to show that $\mathring{\mathcal{H}}_0\Lambda^k(\mathbb{R}^{n+1}) = 0$ when $k = n$.

For any $\alpha \in \mathring{\mathcal{H}}_0\Lambda^n(\mathbb{R}^{n+1})$, we can write it as $\alpha = \sum_i a_i \star_{\mathbb{R}^{n+1}} d\lambda_i$, where the a_i are constants. By assumption α vanishes when pulled back to every hyperplane $\{\lambda_i = 0\}$, so we investigate what happens to the terms in the right-hand side under this restriction. The restriction of $\star_{\mathbb{R}^{n+1}} d\lambda_i$ is nonzero; it is just the volume form on this hyperplane, which we can denote μ_i. On the other hand, for $j \neq i$, the restriction of $\star_{\mathbb{R}^{n+1}} d\lambda_j$ is zero because $\star_{\mathbb{R}^{n+1}} d\lambda_j$ is a wedge product of n factors including $d\lambda_i$. So, since α has vanishing trace, pulling back the equation $\alpha = \sum_i a_i \star_{\mathbb{R}^{n+1}} d\lambda_i$ to the hyperplane yields $0 = a_i\mu_i$, so $a_i = 0$. This argument holds for all i, so $\alpha = 0$. □

5. Finite element spaces

5.1. **Dimensions of the spaces.** We will now compute the dimension of the space $\mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n)$. Recall that this space consists of trace-free (p,q)-forms on T^n that have constant coefficients and belong to the eigenspace of s^*s corresponding to the eigenvalue $m(m+p-q+1)$.

Lemma 5.1. *Assume $0 \leq q \leq p \leq n$ and $0 \leq m \leq q-1$. Then*

$$\dim \mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n) + \dim \mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^{n+1}) = \dim \mathring{\mathcal{H}}_0\Lambda_m^{p,q}(\mathbb{R}^{n+1}).$$

Proof. Taking $r = 0$ in Theorem 4.26 tells us that as long as $m \neq q$, every member of $\mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n)$ admits an extension to $\mathring{\mathcal{H}}_0\Lambda_m^{p,q}(\mathbb{R}^{n+1})$. Put another way, the map

$$\mathrm{Tr}_{T^n} : \mathring{\mathcal{H}}_0\Lambda_m^{p,q}(\mathbb{R}^{n+1}) \to \mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n)$$

which takes each member of $\mathring{\mathcal{H}}_0\Lambda_m^{p,q}(\mathbb{R}^{n+1})$ to its trace on T^n is surjective when $m \neq q$. It follows that

$$\dim \mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n) + \dim\ker \mathrm{Tr}_{T^n} = \dim \mathring{\mathcal{H}}_0\Lambda_m^{p,q}(\mathbb{R}^{n+1}), \quad m \neq q.$$

The kernel of Tr_{T^n} consists of those double forms in $\mathring{\mathcal{H}}_0\Lambda_m^{p,q}(\mathbb{R}^{n+1})$ that have vanishing trace on the coordinate hyperplanes as well as on the hyperplane containing T^n. Equivalently, they have vanishing trace on the boundary of the $(n+1)$-simplex

$$K^{n+1} := \left\{(\lambda_0, \dots, \lambda_n) \mid \lambda_i \geq 0, \sum_i \lambda_i \leq 1\right\}.$$

Since K^{n+1} is isomorphic to T^{n+1} via an affine transformation, it follows from Proposition 3.4 that the kernel of Tr_{T^n} is isomorphic to $\mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^{n+1})$. Thus,

$$\dim \mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n) + \dim \mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^{n+1}) = \dim \mathring{\mathcal{H}}_0\Lambda_m^{p,q}(\mathbb{R}^{n+1}), \quad m \neq q.$$

□

The lemma above provides a recursive formula that we can use to compute the dimension of $\mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n)$. To use it, we will first need to compute the dimensions of $\mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1})$, $\mathring{\mathcal{H}}_0\Lambda_m^{p,q}(\mathbb{R}^{n+1})$, and (to handle the base case $n = p$) $\mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^p)$.

Lemma 5.2. *The dimension of $\mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1})$ is*

$$\dim \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}) = \binom{n+1}{p}\binom{p}{n+1-q} = \binom{n+1}{q}\binom{q}{n+1-p}.$$

Proof. Using Notation 2.3, let $e^i = d\lambda^i$ and let

$$\varphi = \sum_{I,J} c_{I,J} e^{I,J}$$

be an arbitrary (p,q)-form on $\mathbb{R}^{n+1}$ with constant coefficients. We assume that the multi-indices $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are each in increasing order. If we take the trace on a coordinate hyperplane $\{\lambda_i = 0\}$, then every term in this sum has vanishing trace except for the terms with $i \notin I \cup J$. Those terms with $i \notin I \cup J$ are linearly independent (p,q)-forms on $\{\lambda_i = 0\}$, so the corresponding coefficients $c_{I,J}$ with $i \notin I \cup J$ must vanish if φ belongs to $\mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1})$. Therefore

$$\varphi = \sum_{I,J\colon I\cup J=\{0,1,\dots,n\}} c_{I,J} e^{I,J}$$

if $\varphi \in \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1})$. Conversely, every φ of this form clearly has vanishing trace on the coordinate hyperplanes. It follows that the set

$$\{e^{I,J} \mid I \cup J = \{0,1,\dots,n\},\ |I| = p,\ |J| = q\}$$

forms a basis for $\mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1})$, where, once again, the multi-indices above are understood to be in increasing order. Each member $e^{I,J}$ of this basis satisfies $|I \cap J| = p + q - n - 1$. Such an $e^{I,J}$ is formed by choosing p of the integers $\{0,1,2,\dots,n\}$ to go into I and choosing $|I \cap J|$ of those already selected to go into both I and J; the remaining integers in J are then uniquely determined by the condition that $I \cup J = \{0,1,\dots,n\}$. Therefore the space has dimension

$$\dim \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}) = \binom{n+1}{p}\binom{p}{p+q-n-1} = \binom{n+1}{p}\binom{p}{n+1-q}.$$

□

To compute the dimension of the subspace $\mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^{n+1}) \subseteq \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1})$, we introduce some notation.

Notation 5.3. Let $\mathring{s}$ and $\mathring{s}^*$ denote the restrictions of $s : \Lambda^{p,q}(\mathbb{R}^{n+1}) \to \Lambda^{p+1,q-1}(\mathbb{R}^{n+1})$ and $s^* : \Lambda^{p,q}(\mathbb{R}^{n+1}) \to \Lambda^{p-1,q+1}(\mathbb{R}^{n+1})$ to $\mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1})$. Recall from Proposition 3.4 that s and s^* commute with pullbacks, so $s\varphi$ and $s^*\varphi$ have vanishing trace on the coordinate hyperplanes whenever φ does. Thus, we have maps

$$\mathring{s} : \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}) \to \mathring{\mathcal{H}}_0\Lambda^{p+1,q-1}(\mathbb{R}^{n+1})$$

and

$$\mathring{s}^* : \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}) \to \mathring{\mathcal{H}}_0\Lambda^{p-1,q+1}(\mathbb{R}^{n+1}).$$

Lemma 5.4. *Assume $0 \le p,q \le n$ and $m \ge 0$. The operator $\mathring{s}^m : \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}) \to \mathring{\mathcal{H}}_0\Lambda^{p+m,q-m}(\mathbb{R}^{n+1})$ is injective if $p < q - m + 1$ and surjective if $p \ge q - m$.*

Proof. Since $\mathring{s}$ is the restriction of s to a subspace, the map $\mathring{s}^m$ is injective whenever s^m is injective. Similarly, since $\mathring{s}^*$ is the restriction of s^* to a subspace, the map $(\mathring{s}^*)^m$ is injective whenever $(s^*)^m$ is injective, and therefore $\mathring{s}^m$ is surjective whenever s^m is surjective. The conclusion thus follows from Proposition 2.29. ∎

Lemma 5.5. *Assume $0 \le q \le p$ and $0 \le m \le q$. Then*

$$\mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^{n+1}) = \ker \mathring{s}^{m+1} \cap \operatorname{im}(\mathring{s}^*)^m.$$

Proof. Recall from Proposition 2.28 that

$$\Lambda^{p,q}_m(\mathbb{R}^{n+1}) = \ker s^{m+1} \cap \operatorname{im}(s^*)^m.$$

Now, if $\varphi \in \ker \mathring{s}^{m+1} \cap \operatorname{im}(\mathring{s}^*)^m$, then $s^{m+1}\varphi = \mathring{s}^{m+1}\varphi = 0$ and $\varphi = (\mathring{s}^*)^m\psi = (s^*)^m\psi$ for some ψ, so $\varphi \in \ker s^{m+1} \cap \operatorname{im}(s^*)^m \cap \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}) = \mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^{n+1})$.

Conversely, if $\varphi \in \mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^{n+1})$, then φ has vanishing trace on the coordinate hyperplanes and belongs to the kernel of s^{m+1} and the image of $(s^*)^m$, so it belongs to the kernel of $\mathring{s}^{m+1}$ and satisfies $\varphi = (s^*)^m\psi$ for some ψ. We will show that ψ has vanishing trace on the coordinate hyperplanes. Let Tr denote the map that sends double forms on $\mathbb{R}^{n+1}$ to their trace on the union of the coordinate hyperplanes. Since taking the trace commutes with s^*, we have

$$0 = \operatorname{Tr}\varphi = (s^*)^m \operatorname{Tr}\psi.$$

By Lemma 5.4, $(s^*)^m : \Lambda^{p+m,q-m}(\mathbb{R}^{n+1}) \to \Lambda^{p,q}(\mathbb{R}^{n+1})$ is injective, so $\operatorname{Tr}\psi = 0$. It follows that $\varphi \in \ker \mathring{s}^{m+1} \cap \operatorname{im}(\mathring{s}^*)^m$. ∎

Lemma 5.6. *Assume $0 \le q \le p$ and $0 \le m \le q$. The dimension of $\mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^{n+1})$ is*

$$\begin{aligned}
&\dim \mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^{n+1}) \\
&\quad = \binom{n+1}{q-m}\binom{q-m}{n+1-p-m} - \binom{n+1}{q-m-1}\binom{q-m-1}{n-p-m} \\
&\quad = \begin{cases} \dfrac{p-q+2m+1}{q-m}\dbinom{n+1}{q-m-1}\dbinom{q-m}{n+1-p-m}, & \textit{if } m < q, \\ 1, & \textit{if } m = q \textit{ and } \\ & \quad p+q = n+1, \\ 0, & \textit{otherwise.} \end{cases}
\end{aligned}$$

Proof. Since $0 \le q \le p$, Lemma 5.4 implies that $\mathring{s}^{m+1} : \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}) \to \mathring{\mathcal{H}}_0\Lambda^{p+m+1,q-m-1}(\mathbb{R}^{n+1})$ is surjective and $(\mathring{s}^*)^m : \mathring{\mathcal{H}}_0\Lambda^{p+m,q-m}(\mathbb{R}^{n+1}) \to \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1})$ is injective. Therefore

$$\dim \ker \mathring{s}^{m+1} = \dim \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}) - \dim \mathring{\mathcal{H}}_0\Lambda^{p+m+1,q-m-1}(\mathbb{R}^{n+1})$$

and

$$\dim \operatorname{im}(\mathring{s}^*)^m = \dim \mathring{\mathcal{H}}_0\Lambda^{p+m,q-m}(\mathbb{R}^{n+1}).$$

Also, $\operatorname{im}(\mathring{s}^*)^m$ is the orthogonal complement of $\ker \mathring{s}^m \subseteq \ker \mathring{s}^{m+1}$, so

$$\dim(\ker \mathring{s}^{m+1} + \operatorname{im}(\mathring{s}^*)^m) = \dim \mathring{\mathcal{H}}_0\Lambda^{p,q}(\mathbb{R}^{n+1}).$$

Thus,

$$\begin{aligned}
&\dim \mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^{n+1}) \\
&= \dim\ker \mathring{s}^{m+1} + \dim\operatorname{im}(\mathring{s}^*)^m - \dim(\ker \mathring{s}^{m+1} + \operatorname{im}(\mathring{s}^*)^m) \\
&= \dim \mathring{\mathcal{H}}_0\Lambda^{p+m,q-m}(\mathbb{R}^{n+1}) - \dim \mathring{\mathcal{H}}_0\Lambda^{p+m+1,q-m-1}(\mathbb{R}^{n+1}).
\end{aligned}$$

The result then follows from Lemma 5.2. □

Lemma 5.7. *Assume $0 \le q \le p$ and $0 \le m \le q$. The dimension of $\mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^p)$ is*

$$\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^p) = \begin{cases} \binom{p}{q} & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

Proof. When $q \le p$, the only nontrivial (p,q)-forms on T^p are of the form $\omega \otimes \alpha$, where ω is the volume p-form and α is an arbitrary q-form. These double forms belong to the kernel of s and have vanishing trace on the boundary of T^p; hence they belong to $\mathring{\mathcal{P}}_0\Lambda^{p,q}_0(T^p)$ by Proposition 2.28. Since the space of constant q-forms on T^p has dimension $\binom{p}{q}$, the result follows. □

Proposition 5.8. *Assume $0 \le q \le p \le n$ and $0 \le m \le q-1$. Then the dimension of $\mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^n)$ is*

$$\begin{aligned}
\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^n) &= \binom{n+1}{q-m}\binom{q-m-1}{p+q-n-1} - \binom{n+1}{p+m+1}\binom{p+m}{p+q-n-1} \\
&= \frac{p-q+2m+1}{p+m+1}\binom{n+1}{q-m}\binom{q-m-1}{n-p-m}.
\end{aligned}$$

Remark 5.9. The number above has a combinatorial interpretation: It counts the number of standard Young tableaux associated with the partition

$$(n+1-q+m,\, n+1-p-m,\, \underbrace{1, 1, \dots, 1}_{p+q-n-1})$$

of $n+1$.

Proof. We use induction on n. In the base case $n = p$, the formula above gives zero when $m > 0$ and gives

$$\frac{p-q+1}{p+1}\binom{p+1}{q}\binom{q-1}{0} = \binom{p}{q}$$

when $m = 0$, in agreement with Lemma 5.7. Now let $n > p$ and assume the formula holds for $n-1$. Using Lemma 5.1 and denoting $A := p-q+2m+1$, we compute

$$
\begin{aligned}
&\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^n)\\
&= \dim \mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^n) - \dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^{n-1})\\
&= \frac{A}{q-m}\binom{n}{q-m-1}\binom{q-m}{n-p-m} - \frac{A}{p+m+1}\binom{n}{q-m}\binom{q-m-1}{n-p-m-1}\\
&= \frac{A}{n+1}\binom{n+1}{q-m}\frac{q-m}{p+q-n}\binom{q-m-1}{n-p-m}\\
&\quad - \frac{A}{p+m+1}\frac{n+1-q+m}{n+1}\binom{n+1}{q-m}\frac{n-p-m}{p+q-n}\binom{q-m-1}{n-p-m}.
\end{aligned}
$$

Since

$$
q-m-\frac{(n+1-q+m)(n-p-m)}{p+m+1} = \frac{(n+1)(p+q-n)}{p+m+1},
$$

the expression above simplifies to

$$
\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^n) = \frac{A}{p+m+1}\binom{n+1}{q-m}\binom{q-m-1}{n-p-m}.
$$

□

Now that we have determined the dimensions of the trace-free spaces, we know how many degrees of freedom to assign to each subsimplex $f \subseteq T^n$ when constructing our finite element space on T^n. Namely, we assign $\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^l)$ degrees of freedom to f, where $l = \dim f$ and the formula for $\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^l)$ is given in Proposition 5.8. We can also verify that the total number of degrees of freedom associated with all of the subsimplices of T^n matches the dimension of $\mathcal{P}_0\Lambda^{p,q}_m(T^n)$. We know this must be true from the preceding theory, but it is illuminating to verify it with a direct calculation. We begin with a lemma.

Lemma 5.10. *Assume $0 \le q \le p$ and $0 \le m \le q$. The dimension of $\mathcal{P}_0\Lambda^{p,q}_m(T^n)$ is*

$$
\begin{aligned}
\dim \mathcal{P}_0\Lambda^{p,q}_m(T^n) &= \binom{n}{q-m}\binom{n}{p+m} - \binom{n}{q-m-1}\binom{n}{p+m+1}\\
&= \frac{p-q+2m+1}{p+m+1}\binom{n+1}{q-m}\binom{n}{p+m}.
\end{aligned}
$$

Proof. By Proposition 2.28, we have $\mathcal{P}_0\Lambda^{p,q}_m(T^n) = \ker s^{m+1} \cap \operatorname{im}(s^*)^m$, so we can compute its dimension using the same strategy that we used to compute the dimension of $\mathring{\mathcal{H}}_0\Lambda^{p,q}_m(\mathbb{R}^{n+1})$ in the proof of Proposition 5.6. This time $s^{m+1} : \mathcal{P}_0\Lambda^{p,q}(T^n) \to \mathcal{P}_0\Lambda^{p+m+1,q-m-1}(T^n)$ is surjective and $(s^*)^m : \mathcal{P}_0\Lambda^{p+m,q-m}(T^n) \to \mathcal{P}_0\Lambda^{p,q}(T^n)$ is injective, so

$$
\dim \ker s^{m+1} = \dim \mathcal{P}_0\Lambda^{p,q}(T^n) - \dim \mathcal{P}_0\Lambda^{p+m+1,q-m-1}(T^n)
$$

and

$$
\dim \operatorname{im}(s^*)^m = \dim \mathcal{P}_0\Lambda^{p+m,q-m}(T^n).
$$

Also, $\ker s^{m+1} + \operatorname{im}(s^*)^m = \mathcal{P}_0\Lambda^{p,q}(T^n)$, so

$$\begin{aligned}\dim \mathcal{P}_0\Lambda^{p,q}_m(T^n) &= \dim\ker s^{m+1} + \dim\operatorname{im}(s^*)^m - \dim(\ker s^{m+1} + \operatorname{im}(s^*)^m)\\ &= \dim \mathcal{P}_0\Lambda^{p+m,q-m}(T^n) - \dim \mathcal{P}_0\Lambda^{p+m+1,q-m-1}(T^n)\\ &= \binom{n}{p+m}\binom{n}{q-m} - \binom{n}{p+m+1}\binom{n}{q-m-1}.\end{aligned}$$

□

Lemma 5.11. *We have*

$$\sum_{l=0}^{n}\binom{n+1}{l+1}\binom{l+1}{q-m}\binom{q-m-1}{l-p-m} = \binom{n+1}{q-m}\binom{n}{p+m}.$$

Proof. Since

$$\begin{aligned}\binom{n+1}{l+1}\binom{l+1}{q-m} &= \binom{n+1}{q-m}\binom{n+1-q+m}{l+1-q+m}\\ &= \binom{n+1}{q-m}\binom{n+1-q+m}{n-l},\end{aligned}$$

it is enough to show that

$$\sum_{l=0}^{n}\binom{n+1-q+m}{n-l}\binom{q-m-1}{l-p-m} = \binom{n}{p+m}.$$

Equivalently, letting $j = l-p-m$, we must show that

$$\sum_{j=0}^{n-p-m}\binom{n+1-q+m}{n-p-m-j}\binom{q-m-1}{j} = \binom{n}{n-p-m}.$$

This holds because of Vandermonde's identity $\sum_{j=0}^{a}\binom{b}{a-j}\binom{c}{j} = \binom{b+c}{a}$. □

Proposition 5.12. *Assume* $0 \le q \le p \le n$ *and* $0 \le m \le q-1$. *Then*

$$\sum_{l=0}^{n}\binom{n+1}{l+1}\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^l) = \dim \mathcal{P}_0\Lambda^{p,q}_m(T^n).$$

Proof. We use Lemma 5.11 together with the formula for $\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^l)$ given in Proposition 5.8 to compute

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n+1}{l+1}\dim \mathring{\mathcal{P}}_0\Lambda^{p,q}_m(T^l)\\ &\quad= \frac{p-q+2m+1}{p+m+1}\sum_{l=0}^{n}\binom{n+1}{l+1}\binom{l+1}{q-m}\binom{q-m-1}{l-p-m}\\ &\quad= \frac{p-q+2m+1}{p+m+1}\binom{n+1}{q-m}\binom{n}{p+m}.\end{aligned}$$

By Lemma 5.10, this matches the dimension of $\mathcal{P}_0\Lambda^{p,q}_m(T^n)$. □

	n						
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$		1					
$\Lambda_0^{2,1}$			2				
$\Lambda_0^{2,2}$			1	2			
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$				3			
$\Lambda_0^{3,2}$				3	5		
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$					4		
$\Lambda_0^{3,3}$				1	5	5	
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$					6	9	
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$						5	

TABLE 1. Dimension of $\mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n)$ for various values of p, q, m, and n. Zero entries are left blank.

5.2. **Examples of finite element spaces.** We are now ready to discuss the finite element spaces produced by our construction. For the reader's convenience, we list the values of $\dim \mathring{\mathcal{P}}_0\Lambda_m^{p,q}(T^n)$ for various values of p, q, m, and n in Table 1.

5.2.1. *The case* $(p, q) = (1, 1)$. As discussed in Section 2.4.2, the space $\Lambda^{1,1}$ decomposes into two spaces: a space $\Lambda_0^{1,1}$ consisting of symmetric bilinear forms, and a space $\Lambda_1^{1,1}$ consisting of skew-symmetric bilinear forms, i.e. 2-forms. In dimension $n \geq 3$, the latter space does not admit a piecewise constant discretization, and correspondingly our construction fails to produce one because $m = q = 1$. The space $\Lambda_0^{1,1}$, on the other hand, admits a piecewise constant discretization. Referring to the first row of Table 1, the corresponding finite element space has 1 degree of freedom per edge. The elements of this space have single-valued trace on every codimension-1 simplex f, which is equivalent to saying that $\varphi(X; Y)$ is single-valued on f for all vectors X, Y that are tangent to f. This space is the lowest order Regge finite element space studied by Christiansen [8, 9] and Li [21].

5.2.2. *The case* $(p, q) = (2, 1)$. As discussed in Section 2.4.3, the space $\Lambda^{2,1}$ decomposes into two spaces which, in dimension $n = 3$, can be identified with matrices.

The first space, $\Lambda_0^{2,1}$, consists of trace-free matrices under this identification. Our construction yields a piecewise constant finite element space for such trace-free matrices, and, according to Table 1, this finite element space has 2 degrees of freedom per triangle. The matrices in this finite element space have normal-tangential continuity along element interfaces, meaning that $\nu^T A \tau_1$ and $\nu^T A \tau_2$ are single-valued along every triangle f with normal vector ν and tangent basis (τ_1, τ_2). This follows from the identifications between $(2,1)$-forms and matrices discussed in Section 2.4.3. This finite element space coincides with a space introduced by Gopalakrishnan, Lederer, and Schöberl [14].

The members of the second space, $\Lambda_1^{2,1}$, can be identified with multiples of the identity matrix in dimension $n = 3$. Normal-tangential continuity is automatic for such matrices, so there is a trivial finite element space for $\Lambda_1^{2,1}$ in 3D that consists of all piecewise constant multiples of the identity. In dimension $n \geq 4$, $\Lambda_1^{2,1} \simeq \Lambda^3$ fails to admit a piecewise constant discretization with single-valued trace on element interfaces. Correspondingly, our construction fails to produce one since $m = q = 1$.

5.2.3. *The case* $(p, q) = (2, 2)$. As discussed in Section 2.4.4, the space $\Lambda^{2,2}$ decomposes into three spaces. For each space, we will discuss its discretization first in any dimension n and then (if applicable) specialize to $n = 3$.

The first space, $\Lambda_0^{2,2}$, consists of algebraic curvature tensors. Our piecewise constant finite element space for such tensors, which appears to be new (in dimension $n \geq 4$), has 1 degree of freedom per triangle and 2 degrees of freedom per tetrahedron according to the third row of Table 1. The tensors in this finite element space have the property that for every element interface f, $\varphi(X, Y; Z, W)$ is single-valued on f for all vectors X, Y, Z, W that are tangent to f. (The same is true for shared simplices of lower dimension too.) In dimension $n = 3$, we can identify each member of $\Lambda_0^{2,2}$ with a symmetric 3×3 matrix A, and the aforementioned continuity property reduces to the statement that $\nu^T A \nu$ is single-valued, where ν is the unit normal to f. This finite element space in dimension $n = 3$ coincides with a space introduced by Sinwel [25].

The second space, $\Lambda_1^{2,2}$, consists of skew-symmetric $(2,2)$-forms. Its piecewise constant finite element discretization has 3 degrees of freedom per tetrahedron according to Table 1. In dimension $n = 3$, every skew-symmetric $(2,2)$-form can be identified with a skew-symmetric 3×3 matrix A, and the aforementioned continuity property—normal-normal continuity—is vacuous since $\nu^T A \nu$ automatically vanishes. Thus, this finite element space simply consists of all piecewise constant skew-symmetric 3×3 matrices.

The third space, $\Lambda_2^{2,2}$ consists of $(2,2)$-forms that alternate in all 4 arguments; i.e. 4-forms. This space fails to admit a piecewise constant discretization in dimension $n \geq 5$. Correspondingly, our construction fails to produce one since $m = q = 2$.

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