

# Blow-up finite elements

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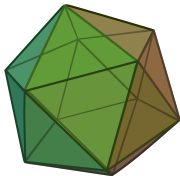
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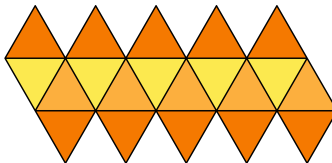
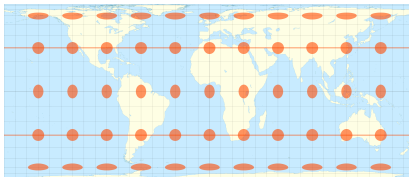


# Surfaces: extrinsic vs. intrinsic

Extrinsic



Intrinsic



## Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

Four images from Wikipedia

# Discretizing continuous tangent vector fields on surfaces is “impossible”

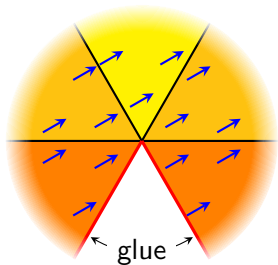
## Vector field requirements

- 1 Tangent to the surface.
- 2 No jump across edges.
- 3 Continuous on each triangle.

## Let's try!

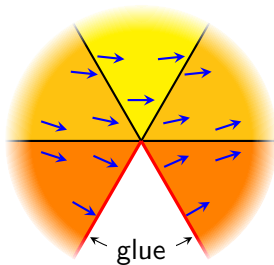
- Zoom in on a vertex of the icosahedron.
- What do we see?

continuous elements



discontinuous across red edge

blow-up elements



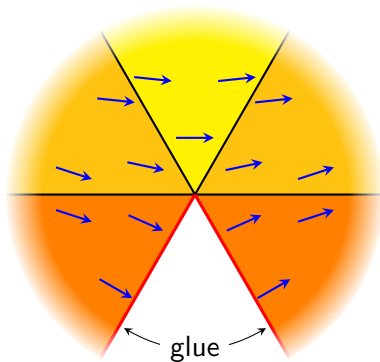
continuous across all edges

# Approaches to discretizing surface vector fields

- Break tangentiality to surface: Use vector fields in ambient space, penalize component normal to surface.
  - Unnecessary data (normal component).
  - Extrinsic.
- Break continuity across edges:
  - For problems like electromagnetism ( $H(\text{curl})$  (or  $H(\text{div}))$ ), use FEEC (BDM, RT) elements.
    - Jump in component normal to edge but not in component tangent to edge (or vice versa).
    - Intrinsic via FEEC (differential forms) formulation.
    - In this context, edge jumps are actually good (full continuity is bad), see (Arnold, Falk, Winther, 2010).
  - In contrast, for problems like fluid flow ( $H^1$ ), edge jumps are bad.
    - One approach: Use FEEC anyways and penalize jump across edges.
    - Demlow, Neilan approach: At each vertex, project to a plane (no angle defect), and enforce continuity in that plane.
- Break continuity at vertices: Blow-up elements (—, Gawlik).
  - Intrinsic.
  - Discontinuities are *only* where the problem is (vertices with angle defect).

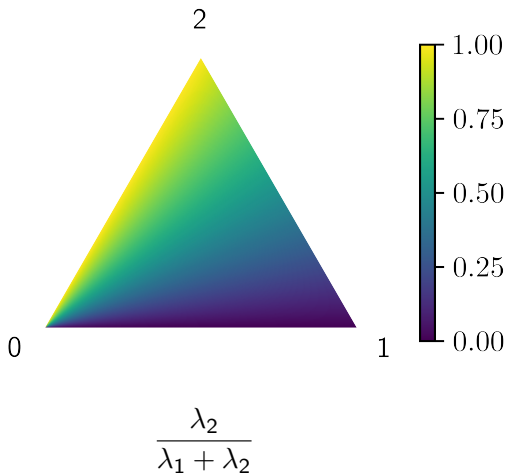
# Break continuity at vertices

blow-up elements

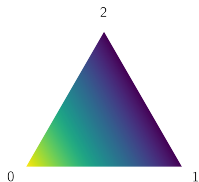


continuous across all edges  
discontinuous at vertices

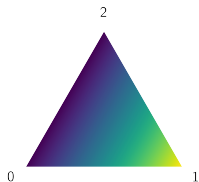
# A simplicial analogue of the angular coordinate



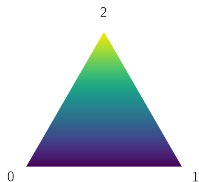
# Lagrange $\mathcal{P}_1$ shape functions



$\lambda_0$

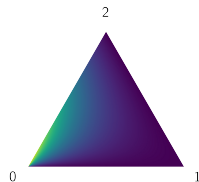
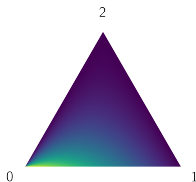


$\lambda_1$

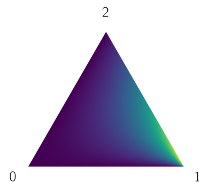
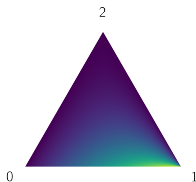


$\lambda_2$

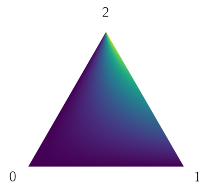
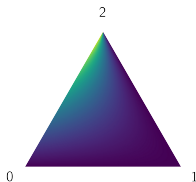
# Blow-up $b\mathcal{P}_1$ shape functions



$$\psi_{012} = \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0 \lambda_2}{\lambda_2 + \lambda_1},$$



$$\psi_{102} = \frac{\lambda_1 \lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_0},$$



$$\psi_{201} = \frac{\lambda_2 \lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2 \lambda_1}{\lambda_1 + \lambda_0}.$$

## Shape function

$$\psi_{012} = \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \frac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_2}{\lambda_2}.$$

## Earlier appearances

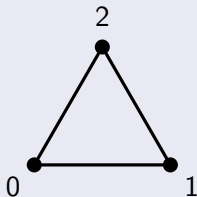
- Geometric invariants (Chen, 1957).
- Horse betting (Harville, 1973).
- Intersection homology (Brasselet, Goresky, MacPherson, 1991; Bendiffalah, 1995).

## Poisson processes (—, Gawlik)

- Three radiation sources with rates  $\lambda_0$ ,  $\lambda_1$ , and  $\lambda_2$ .
- Let  $t_0$ ,  $t_1$ ,  $t_2$  be the times when the respective radiation sources produce their first particle.
- $\psi_{012}$  is the probability that  $t_0 \leq t_1 \leq t_2$ .

# Degrees of freedom

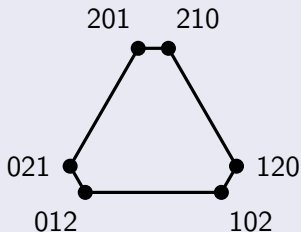
## Classical Lagrange $\mathcal{P}_1$



Barycentric coordinates:  $\lambda_0 + \lambda_1 + \lambda_2 = 1$ .

- 0 :  $\lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- 1 :  $\lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
- 2 :  $\lambda_2 = 1 \Leftrightarrow \lambda_0 = \lambda_1 = 0$

## Blow-up $b\mathcal{P}_1$



- 012 :  $\lim_{\lambda_1 \rightarrow 0} \lim_{\lambda_2 \rightarrow 0}$

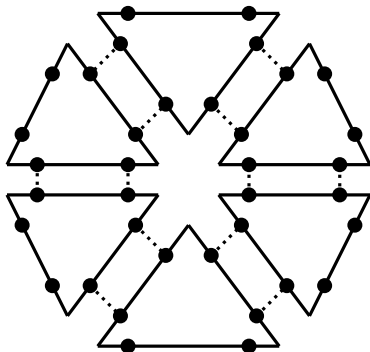
- 120 :  $\lim_{\lambda_2 \rightarrow 0} \lim_{\lambda_0 \rightarrow 0}$

- 201 :  $\lim_{\lambda_0 \rightarrow 0} \lim_{\lambda_1 \rightarrow 0}$

- 021 :  $\lim_{\lambda_2 \rightarrow 0} \lim_{\lambda_1 \rightarrow 0}$

- 102 :  $\lim_{\lambda_0 \rightarrow 0} \lim_{\lambda_2 \rightarrow 0}$

- 210 :  $\lim_{\lambda_1 \rightarrow 0} \lim_{\lambda_0 \rightarrow 0}$



Blow-up finite elements

- Scalar fields: we place a number at each dot.
- Vector fields: we place two numbers at each dot, for the tangential and normal components, respectively.
  - Enforce continuity for **both** components, yielding **full continuity across edges**.
- Matrix fields: At each dot, we record the tangential–tangential component, the tangential–normal component, etc.
  - Can impose conditions on the components such as symmetry, trace-free, etc.
  - Can enforce continuity for all components or just some of them.
- General tensor fields are analogous.

# Vector Laplacian eigenvalue problems on surfaces

## Hodge Laplacian (e.g. Maxwell)

$$(dd^* + d^*d)v^b = \lambda v^b.$$

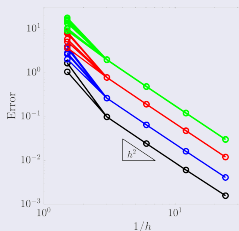
- Tangential continuity across edges suffices.
- Standard FEEC works.
- $L^2$  pairing suffices.

## Bochner Laplacian (e.g. Stokes)

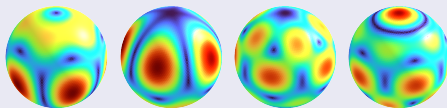
$$\nabla^* \nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

## Bochner Laplacian on sphere using blow-up elements



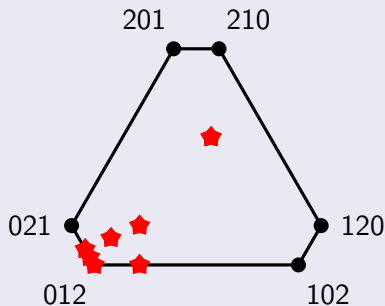
Eigenvalue error



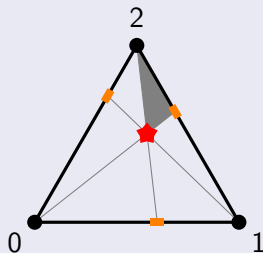
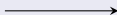
Eigenfield magnitude ( $\lambda = 11, 11, 19, 19$ )

# Blowing up the simplex

## Configuration space of barycentric subdivisions



Smooth



Smooth “in polar coordinates”

- Another interpretation of  $\psi_{012} = \lambda_0 \frac{\lambda_1}{\lambda_1 + \lambda_2}$ : the relative area of the shaded region.
- For the general theory of blowing up manifolds with corners, see Melrose, 1996.

# $H^1$ nonconforming

## Blow-up fields are not in $H^1$

- Near vertices, blow-up scalar fields behave like  $\theta$ .
- $\text{grad } \theta$  grows like  $\frac{1}{r}$ .
- $\frac{1}{r} \in L^{2-\varepsilon}$  (or  $H^{-\varepsilon}$ ) but not in  $L^2$ .

## Impossibility of $H^1$ discretization of tangent vector fields

- Curvature can be computed via second derivatives of vector fields.
- Curvature is delta functions at vertices times angle defect.
- Delta functions are in  $H^{-1-\varepsilon}$  but not in  $H^{-1}$ .

## Bochner Laplacian

- Weak formulation  $\int \langle \nabla u, \nabla v \rangle = \lambda \int \langle u, v \rangle$ .
- $u, v \notin H^1$  means left-hand side is infinite, though it becomes finite if we remove a small neighborhood of the vertices.
- Use quadrature rule that doesn't evaluate at vertices, and it works?

# There's more

## So far in this talk

- Lowest order blow-up scalar fields in two dimensions,  $b\mathcal{P}_1(T^2)$ ,
  - including tensor fields with components in  $b\mathcal{P}_1(T^2)$ .

## Our paper

- **Differential complex** of blow-up Whitney  **$k$ -forms** in **any dimension**,  $b\mathcal{P}_1^-\Lambda^k(T^n)$ .
  - Shape functions previously studied in (Brasselet, Goresky, MacPherson, 1991), called shadow forms.
- ~~Higher-order blow-up scalar fields  $b\mathcal{P}_r(T^n)$ .~~
- A surprising connection to arrival times of Poisson processes, yielding simpler computations.
- Degrees of freedom of blow-up Whitney  $k$ -forms are integration over  $k$ -faces of the blow-up simplex.

# What's next

## Higher order blow-up scalar fields $bP_r$ .

- Fluid flow (surface Stokes complex): blow-up generalization of  $P_2$ – $P_1$  velocity–pressure pair. (See also Demlow, Neilan and Kone, Neilan, Poling for extrinsic method.)
- Implement in NGSolve, joint with students Aidan Nelappana and Daniel Siebel-Cortopassi.

## Connections to other uses of rational functions?

- Rational functions for Stokes problem (Guzmán, Neilan, 2014).
- Duffy transform, Gregory patches, ...

## Distributional Riemann curvature

- Via blow-ups and frames: (Gawlik, McKee, 2026).
- Original approach (—, Gawlik) via blow-ups and

$$\int_M \langle R_{\text{dist}} v, w \rangle \wedge \beta := \int_M \langle \nabla v \wedge \nabla w \rangle \wedge \beta + \int_M \langle \nabla v, w \rangle \wedge d\beta$$



Yakov Berchenko-Kogan and Evan S. Gawlik

Blow-up Whitney forms, shadow forms, and Poisson processes.

Results in Applied Mathematics, special issue on Hilbert complexes,  
Paper No. 100529, 2025.



J. P. Brasselet, M. Goresky, and R. MacPherson.

Simplicial differential forms with poles.

*Amer. J. Math.*, 113(6):1019–1052, 1991.

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