Blow-up finite elements

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Florida Institute of Technology
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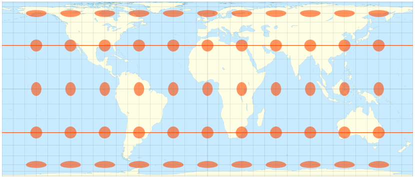
September 8, 2026

Surfaces: extrinsic vs. intrinsic

Extrinsic

Intrinsic



Four images from Wikipedia

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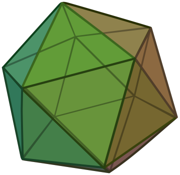
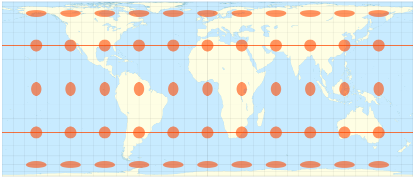
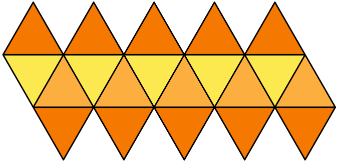


Intrinsic





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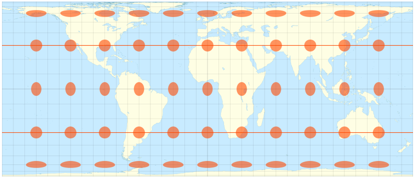

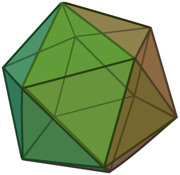

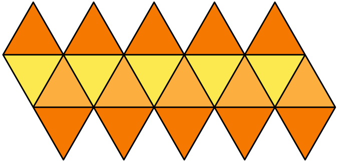

Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

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Discretizing continuous tangent vector fields on surfaces is "impossible"

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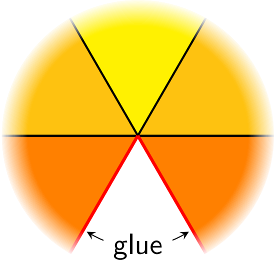

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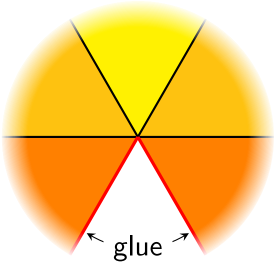

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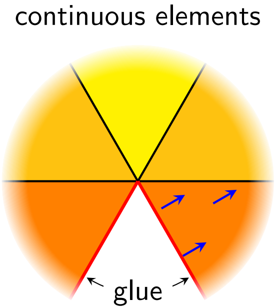

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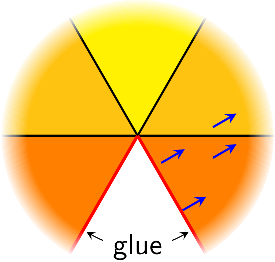

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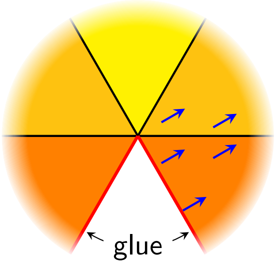

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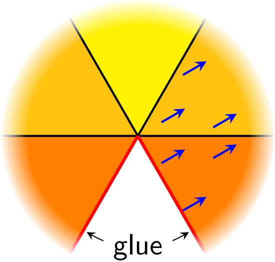

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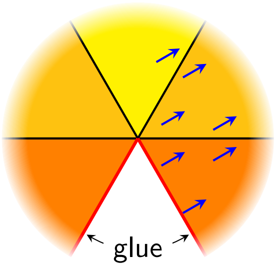

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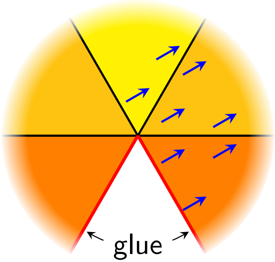

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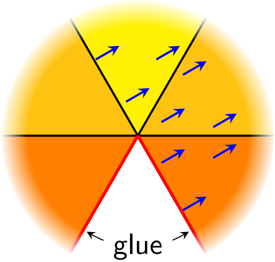

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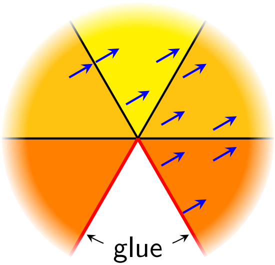

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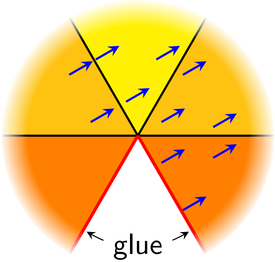

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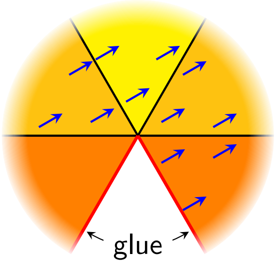

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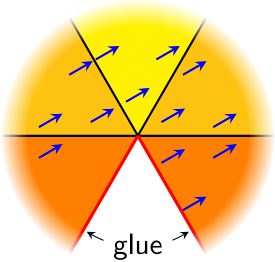

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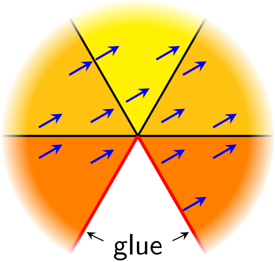

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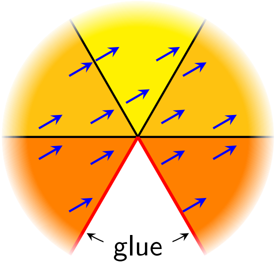

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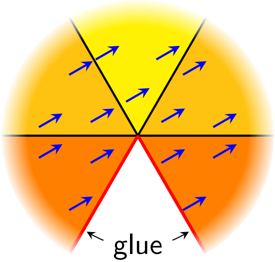

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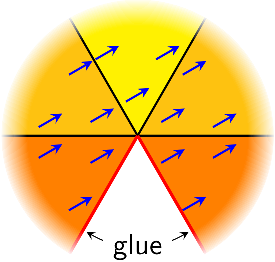

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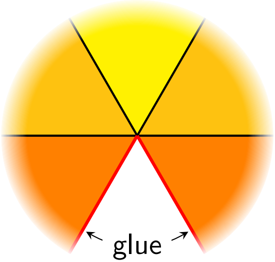

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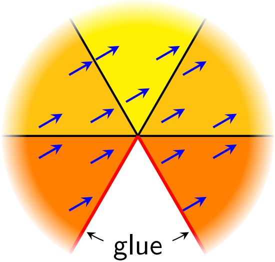

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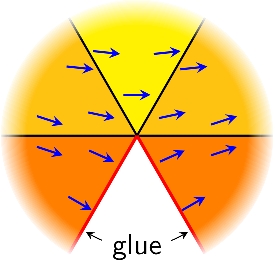

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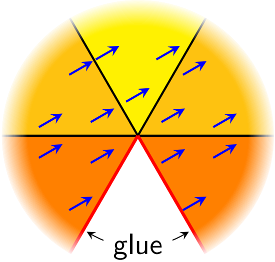

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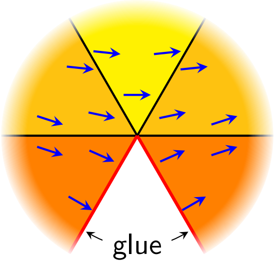

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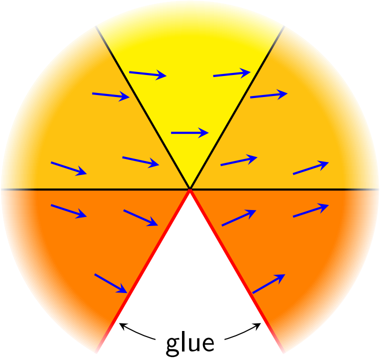
  - Discontinuities are *only* where the problem is (vertices with angle defect).

Break continuity at vertices

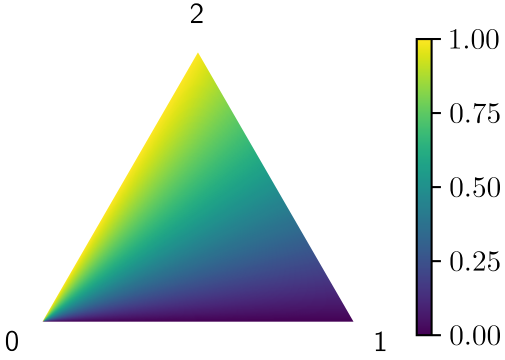
blow-up elements



continuous across all edges

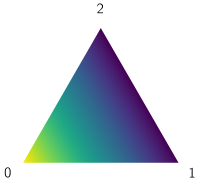
discontinuous at vertices

A simplicial analogue of the angular coordinate

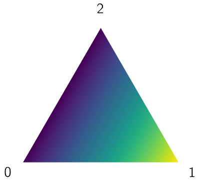


$$\frac{\lambda_2}{\lambda_1 + \lambda_2}$$

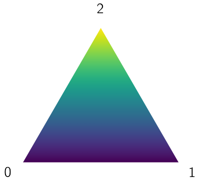
Lagrange $\mathcal{P}_1$ shape functions



λ_0

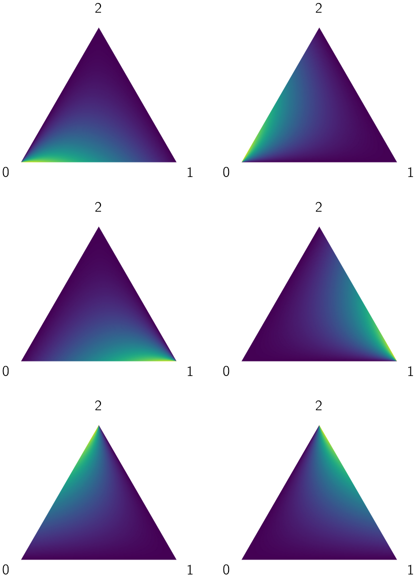


λ_1



λ_2

Blow-up $b\mathcal{P}_1$ shape functions



$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

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$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

New?

Shape function

$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{\lambda_2}{\lambda_2}.$$

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Earlier appearances

New?

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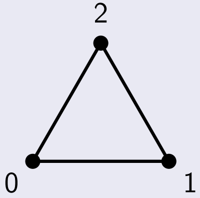
- Let t_0, t_1, t_2 be the times when the respective radiation sources produce their first particle.
- ψ_{012} is the probability that $t_0 \le t_1 \le t_2$.

Degrees of freedom

Classical Lagrange $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
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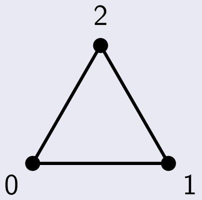

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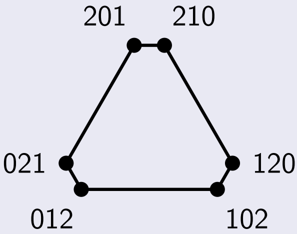
Blow-up $b\mathcal{P}_1$



- $012 : \lim\limits_{\lambda_1\to0} \lim\limits_{\lambda_2\to0}$
- $120 : \lim\limits_{\lambda_2\to0} \lim\limits_{\lambda_0\to0}$
- $201 : \lim\limits_{\lambda_0\to0} \lim\limits_{\lambda_1\to0}$
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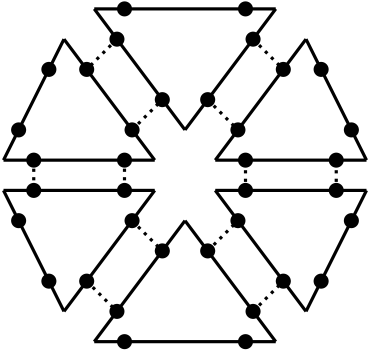
Global spaces



Blow-up finite elements

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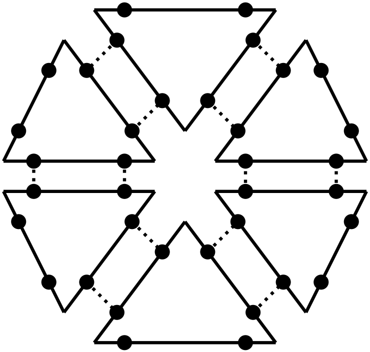
Global spaces



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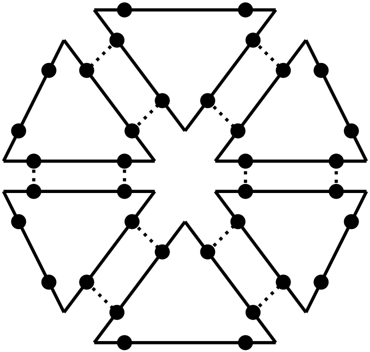

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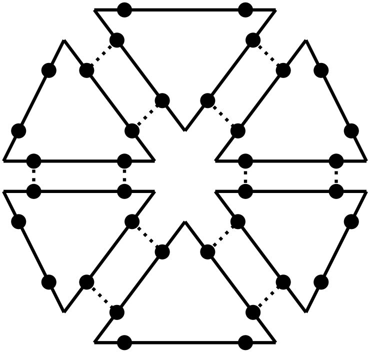

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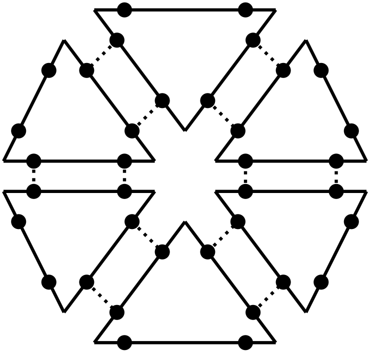

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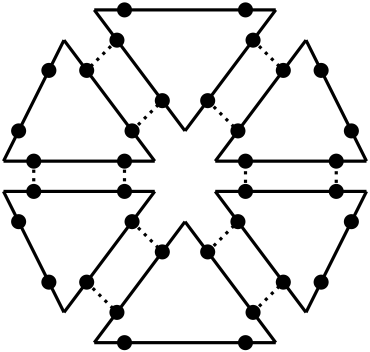

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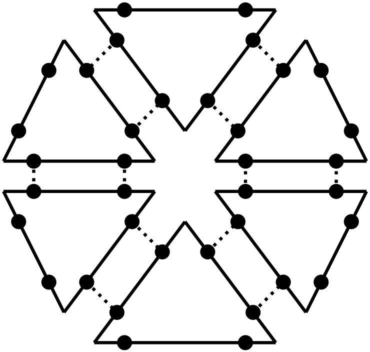

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- General tensor fields are analogous.

Vector Laplacian eigenvalue problems on surfaces

Hodge Laplacian (e.g. Maxwell)

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

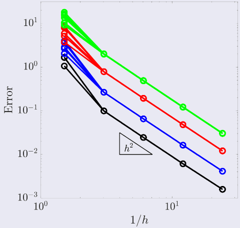
- Tangential continuity across edges suffices.
- Standard FEEC works.
- L^2 pairing suffices.

Bochner Laplacian (e.g. Stokes)

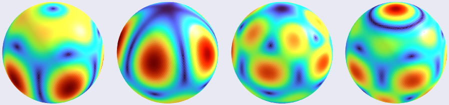
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

Bochner Laplacian on sphere using blow-up elements



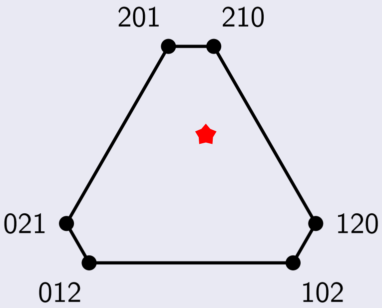
Eigenvalue error



Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

Blowing up the simplex

Configuration space of barycentric subdivisions



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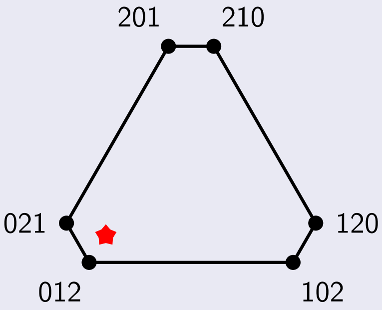

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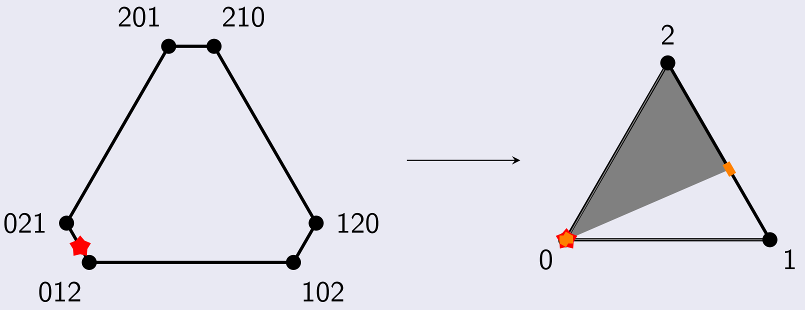

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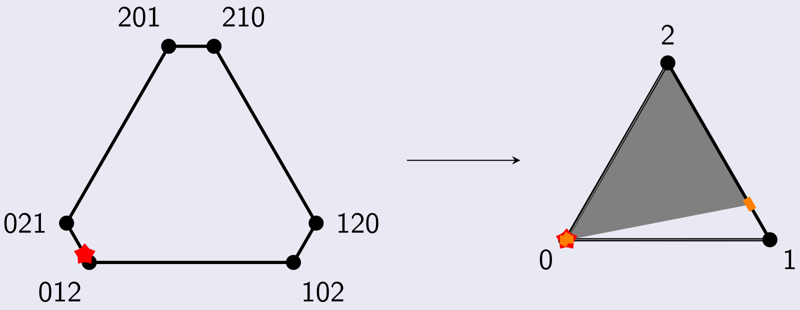

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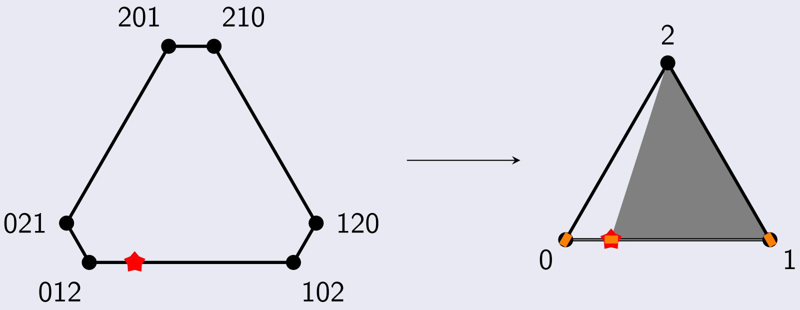

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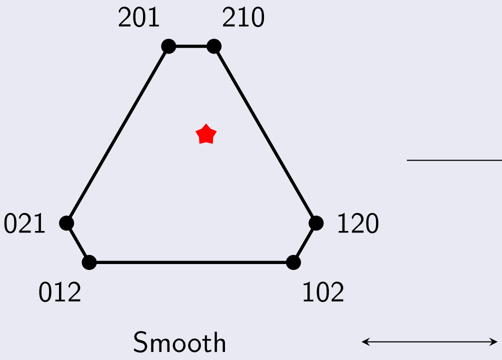

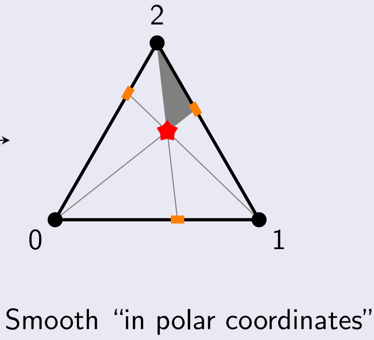

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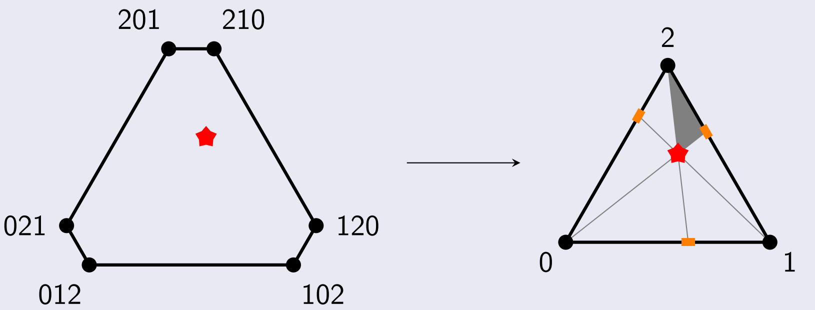

Smooth ⟷ Smooth "in polar coordinates"

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- For the general theory of blowing up manifolds with corners, see Melrose, 1996.

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- Via blow-ups and frames: (Gawlik, McKee, 2026).

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Higher order blow-up scalar fields bP_r.

- Fluid flow (surface Stokes complex): blow-up generalization of P_2–P_1 velocity–presure pair. (See also Demlow, Neilan and Kone, Neilan, Poling for extrinsic method.)
- Implement in NGSolve, joint with students Aidan Nelappana and Daniel Siebel-Cortopassi.

Connections to other uses of rational functions?

- Rational functions for Stokes problem (Guzmán, Neilan, 2014).
- Duffy transform, Gregory patches, ...

Distributional Riemann curvature

- Via blow-ups and frames: (Gawlik, McKee, 2026).
- Original approach (—, Gawlik) via blow-ups and
$$\int_M \langle R_{\mathrm{dist}} v, w \rangle \wedge \beta := \int_M \langle \nabla v \wedge \nabla w \rangle \wedge \beta + \int_M \langle \nabla v, w \rangle \wedge d\beta$$

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik
Blow-up Whitney forms, shadow forms, and Poisson processes.
Results in Applied Mathematics, special issue on Hilbert complexes, Paper No. 100529, 2025.

J. P. Brasselet, M. Goresky, and R. MacPherson.
Simplicial differential forms with poles.
Amer. J. Math., 113(6):1019–1052, 1991.

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