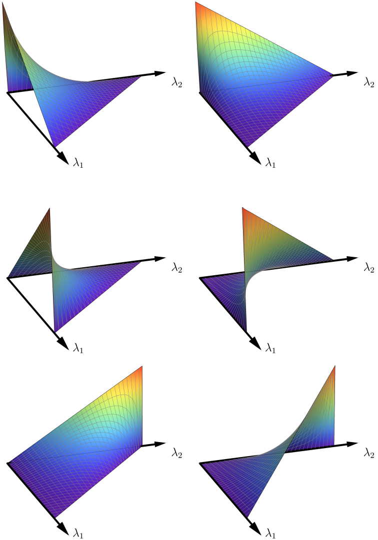
Blow-up Finite Elements

Yakov Berchenko-Kogan, joint with Evan Gawlik

Florida Institute of Technology

April 19–20, 2024

New finite element space



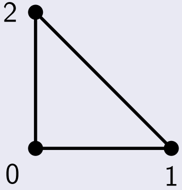
$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

Degrees of freedom

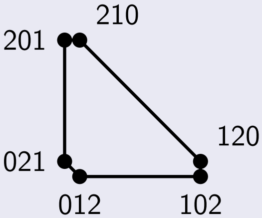
Classical $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
- $2 : \lambda_2 = 1 \Leftrightarrow \lambda_0 = \lambda_1 = 0$

Blow-up $b\mathcal{P}_1$



- $012 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $120 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_0 \to 0}$
- $201 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $021 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $102 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $210 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_0 \to 0}$

Example: Evaluating degrees of freedom

Recall

$$\lambda_0 + \lambda_1 + \lambda_2 = 1, \qquad \psi_{012} = \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2}.$$

Evaluating degrees of freedom

$$012: \lim_{\lambda_1 \to 0} \lim_{\lambda_2 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_1 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1} = \lim_{\lambda_0 \to 1} \lambda_0 = 1,$$

$$021: \lim_{\lambda_2 \to 0} \lim_{\lambda_1 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_2 \to 0} \frac{0}{\lambda_2} = 0,$$

$$120: \lim_{\lambda_2 \to 0} \lim_{\lambda_0 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_2 \to 0} \frac{0}{1} = 0,$$

$$102: \lim_{\lambda_0 \to 0} \lim_{\lambda_2 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_0 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1} = 0,$$

$$201: \lim_{\lambda_0 \to 0} \lim_{\lambda_1 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_0 \to 0} \frac{0}{\lambda_2} = 0,$$

$$210: \lim_{\lambda_1 \to 0} \lim_{\lambda_0 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_1 \to 0} \frac{0}{1} = 0.$$

Motivation

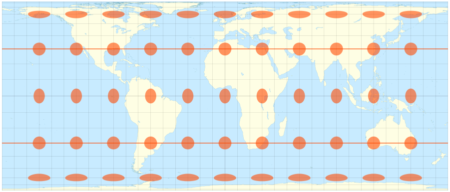
Motivating problem

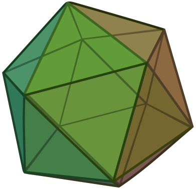
- Goal: construct intrinsic discretizations of tangent vector fields on smooth surfaces that are continuous across edges.
- Obstruction to using classical $\mathcal{P}_1$ elements: angle defect.

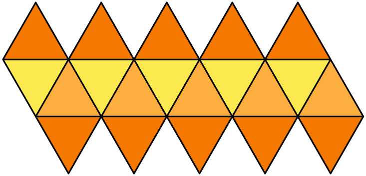
Remark about FEEC

- FEEC discretizations are intrinsic but only tangentially continuous across edges. Normal components are generally discontinuous.
- FEEC discretization suffices for Hodge Laplacian, but not for Bochner Laplacian.

Extrinsic vs. Intrinsic







Why compute intrinsically?

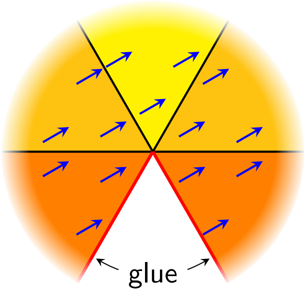
- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

Four images from Wikipedia

Angle defect obstruction to continuous elements

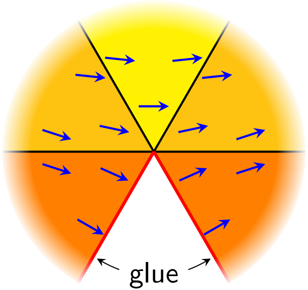
- Try to construct a tangent vector field on the icosahedron.
- What do we see when we zoom in on a vertex?

continuous elements



continuous on each triangle
discontinuous across red edge

blow-up elements



continuous across all edges
discontinuous on each triangle

Vector Laplacian eigenvalue problems

Hodge Laplacian

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

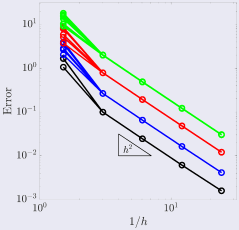
- Tangential continuity suffices.
- Standard FEEC works.

Bochner Laplacian

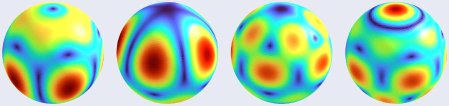
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.

Bochner Laplacian on sphere using blow-up elements



Eigenvalue error



Eigenfield magnitude
$(\lambda = 11, 11, 19, 19)$

There's more

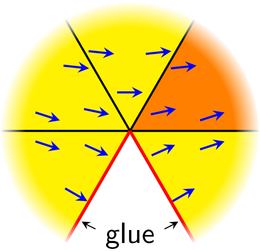
This talk so far

- Lowest order blow-up elements in two dimensions, $b\mathcal{P}_1(T^2)$,
    - including vector fields with components in $b\mathcal{P}_1(T^2)$.

Our preprint

- Differential complex of blow-up Whitney forms, $b\mathcal{P}_1^- \Lambda^k(T^n)$.
    - Shape functions previously studied in (Brasselet, Goresky, MacPherson, 1991), called shadow forms.
- Higher-order blow-up scalar fields $b\mathcal{P}_r(T^n)$.
- A surprising connection to arrival times of Poisson processes, yielding simpler computations.
    - Three radiation sources with rates λ_0, λ_1, and λ_2, sum 1.
    - Let t_0, t_1, t_2 be the times when the respective radiation sources produce their first particle.
    - $\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}$ is the probability that $t_0 \leq t_1 \leq t_2$.
- Degrees of freedom in terms of blow-up simplex.

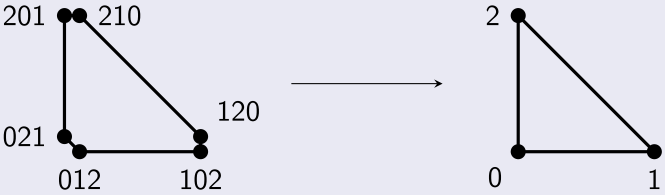
Blowing up



- Even on an individual triangle, the vector field is not continuous at the origin.
- But it is "continuous in polar coordinates," i.e. in r and θ.

Blowing up manifolds with corners (Melrose, 1996)

- formalizes continuity/smoothness "in polar coordinates"



Smooth ⟷ Smooth "in polar coordinates"

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik
Blow-up Whitney forms, shadow forms, and Poisson processes.
https://arxiv.org/abs/2402.03198, 2024.

J. P. Brasselet, M. Goresky, and R. MacPherson.
Simplicial differential forms with poles.
Amer. J. Math., 113(6):1019–1052, 1991.

R. B. Melrose.
Differential analysis on manifolds with corners.
https://math.mit.edu/~rbm/book.html, 1996.