# Blow-up Finite Elements

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Florida Institute of Technology

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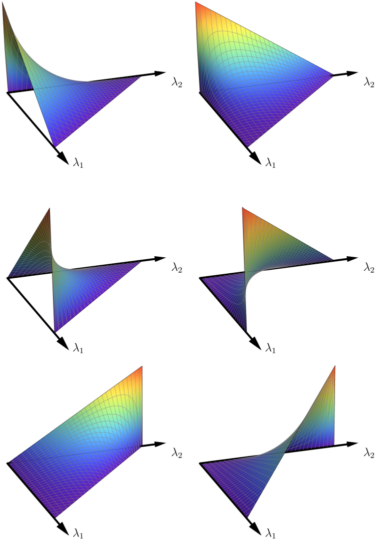
# New finite element space



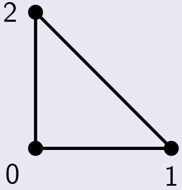
$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2+\lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0+\lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2+\lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0+\lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1+\lambda_0}.$$

# Degrees of freedom

## Classical $\mathcal{P}_1$




Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
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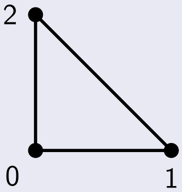


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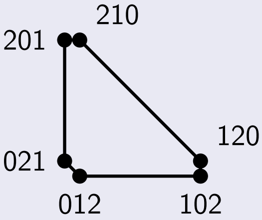
## Blow-up $b\mathcal{P}_1$




- $012 : \lim_{\lambda_1 \to 0} \lim_{\lambda_2 \to 0}$
- $120 : \lim_{\lambda_2 \to 0} \lim_{\lambda_0 \to 0}$
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# Example: Evaluating degrees of freedom

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## Evaluating degrees of freedom

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$$021 : \lim_{\lambda_2\to 0} \lim_{\lambda_1\to 0} \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_2\to 0} \frac{0}{\lambda_2} = 0,$$

$$120 : \lim_{\lambda_2\to 0} \lim_{\lambda_0\to 0} \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_2\to 0} \frac{0}{1} = 0,$$

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## Motivating problem

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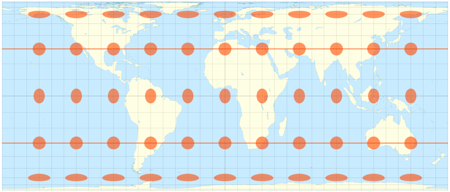
## Remark about FEEC

- FEEC discretizations are intrinsic but only tangentially continuous across edges. Normal components are generally discontinuous.
- FEEC discretization suffices for Hodge Laplacian, but not for Bochner Laplacian.

# Extrinsic vs. Intrinsic



Four images from Wikipedia

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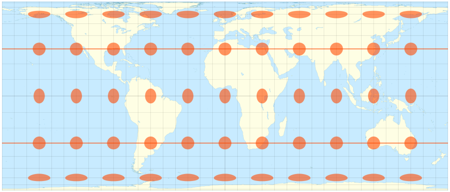

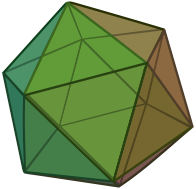

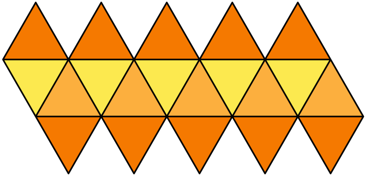

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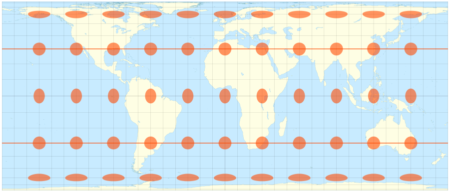

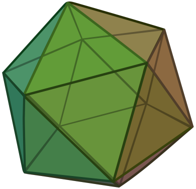

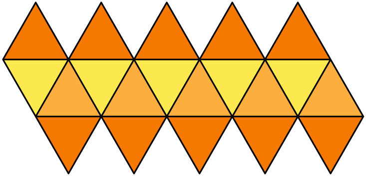

## Why compute intrinsically?

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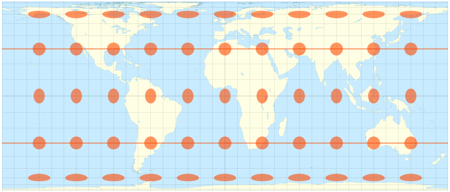

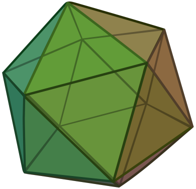

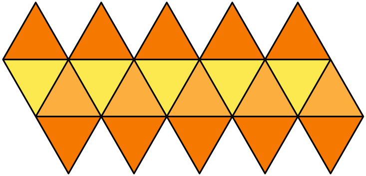

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- Intrinsic problems, e.g. numerical relativity, Ricci flow.

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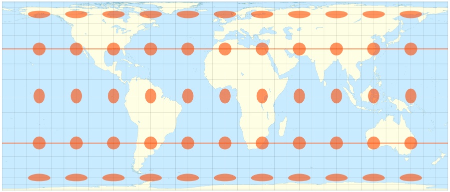

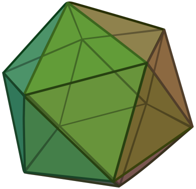

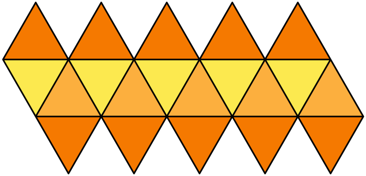

**Why compute intrinsically?**

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

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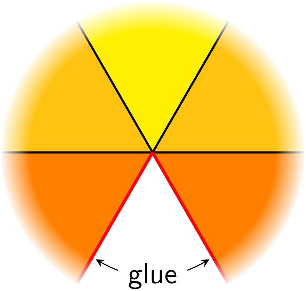

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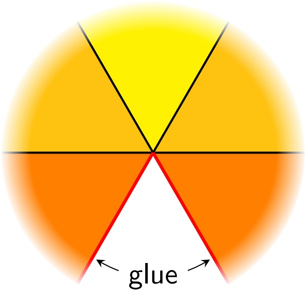

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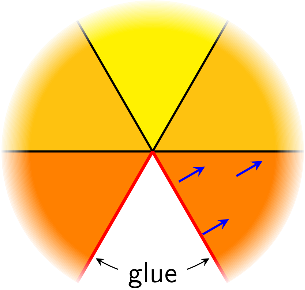

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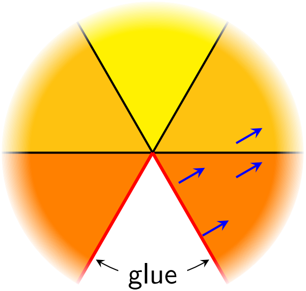

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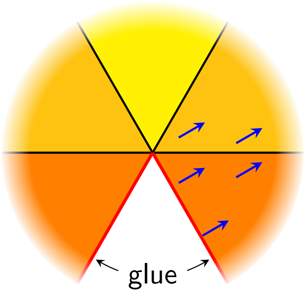

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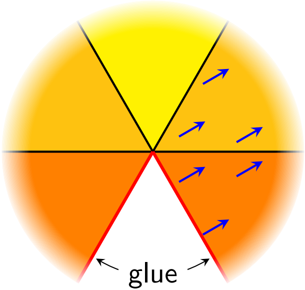

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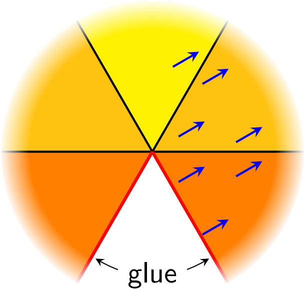

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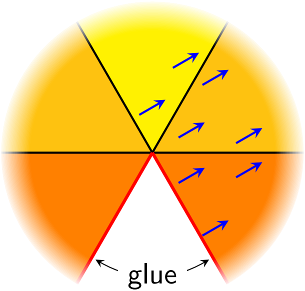

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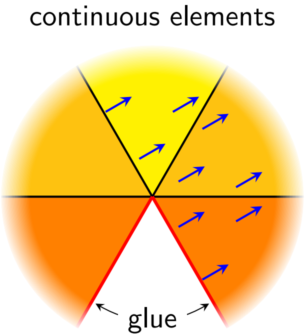

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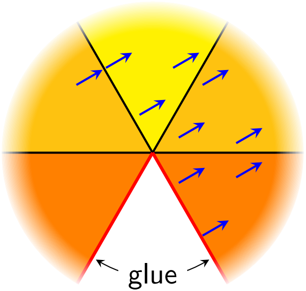

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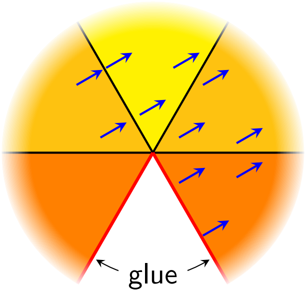

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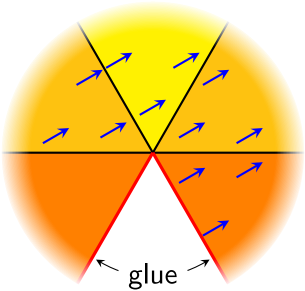

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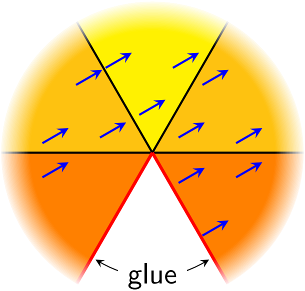

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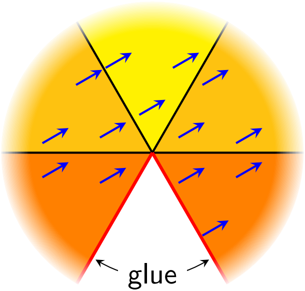

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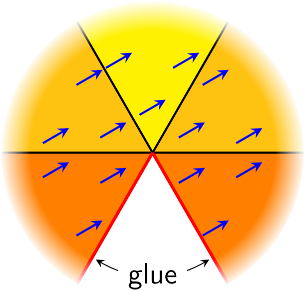

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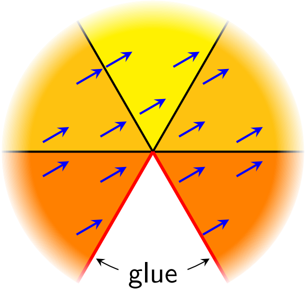


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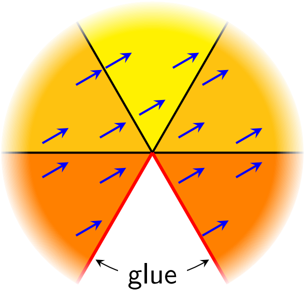


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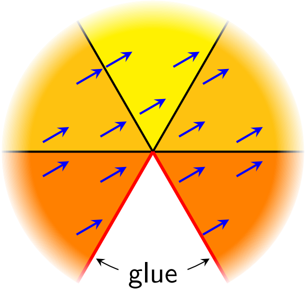


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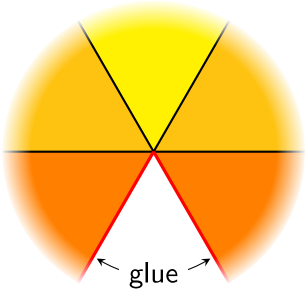

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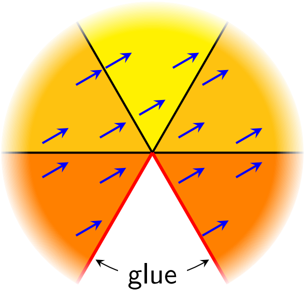


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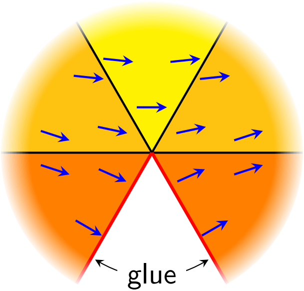

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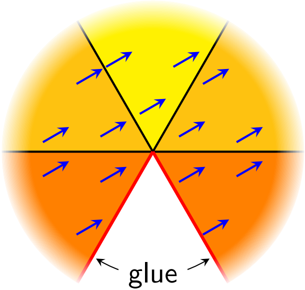


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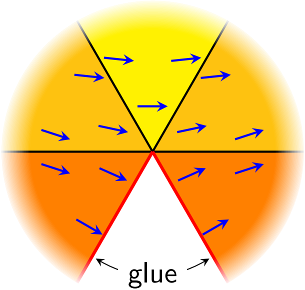


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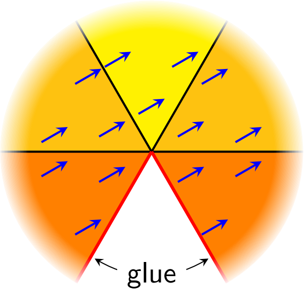


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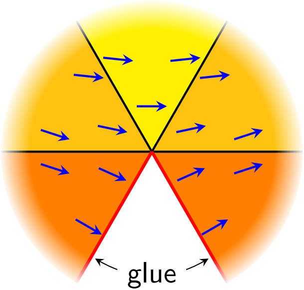


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# Vector Laplacian eigenvalue problems

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$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

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- Standard FEEC works.

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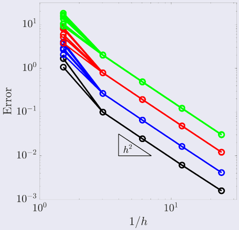
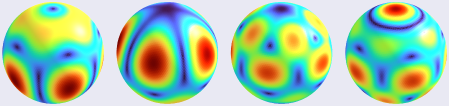
## Bochner Laplacian on sphere using blow-up elements




Eigenvalue error



Eigenfield magnitude
$(\lambda = 11, 11, 19, 19)$

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- Higher-order blow-up scalar fields $b\mathcal{P}_r(T^n)$.

# There's more

## This talk so far

- Lowest order blow-up elements in two dimensions, $b\mathcal{P}_1(T^2)$,
    - including vector fields with components in $b\mathcal{P}_1(T^2)$.

## Our preprint

- Differential complex of blow-up Whitney forms, $b\mathcal{P}_1^-\Lambda^k(T^n)$.
    - Shape functions previously studied in (Brasselet, Goresky, MacPherson, 1991), called shadow forms.
- Higher-order blow-up scalar fields $b\mathcal{P}_r(T^n)$.
- A surprising connection to arrival times of Poisson processes, yielding simpler computations.

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  - Three radiation sources with rates $\lambda_0$, $\lambda_1$, and $\lambda_2$, sum 1.

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  - Let $t_0$, $t_1$, $t_2$ be the times when the respective radiation sources produce their first particle.
  - $\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}$ is the probability that $t_0 \leq t_1 \leq t_2$.

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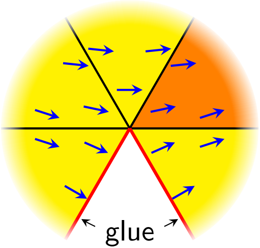
  - $\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}$ is the probability that $t_0 \leq t_1 \leq t_2$.
- Degrees of freedom in terms of blow-up simplex.

# Blowing up




- Even on an individual triangle, the vector field is not continuous at the origin.
- But it is "continuous in polar coordinates," i.e. in $r$ and $\theta$.

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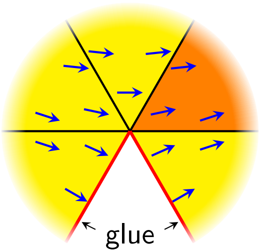


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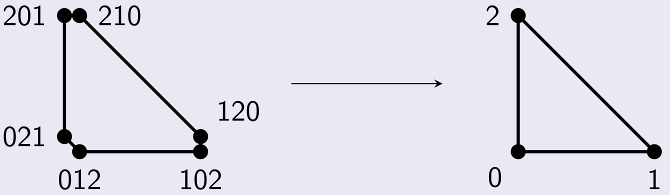
## Blowing up manifolds with corners (Melrose, 1996)

- formalizes continuity/smoothness “in polar coordinates”




Smooth ⟷ Smooth “in polar coordinates”

# Thank you


Yakov Berchenko-Kogan and Evan S. Gawlik
Blow-up Whitney forms, shadow forms, and Poisson processes.
https://arxiv.org/abs/2402.03198, 2024.

J. P. Brasselet, M. Goresky, and R. MacPherson.
Simplicial differential forms with poles.
*Amer. J. Math.*, 113(6):1019–1052, 1991.

R. B. Melrose.
Differential analysis on manifolds with corners.
https://math.mit.edu/~rbm/book.html, 1996.