Blow-up Finite Elements

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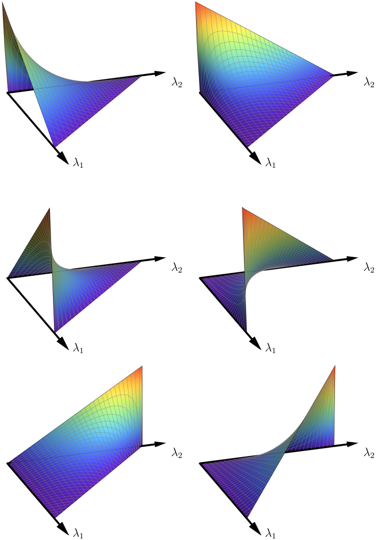
New finite element space



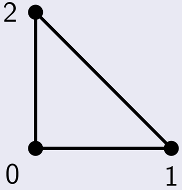
$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

Degrees of freedom

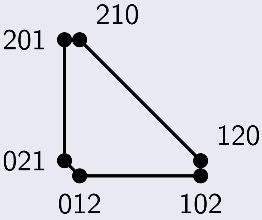
Classical $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
- $2 : \lambda_2 = 1 \Leftrightarrow \lambda_0 = \lambda_1 = 0$

Blow-up $b\mathcal{P}_1$



- $012 : \lim_{\lambda_1 \to 0} \lim_{\lambda_2 \to 0}$
- $120 : \lim_{\lambda_2 \to 0} \lim_{\lambda_0 \to 0}$
- $201 : \lim_{\lambda_0 \to 0} \lim_{\lambda_1 \to 0}$
- $021 : \lim_{\lambda_2 \to 0} \lim_{\lambda_1 \to 0}$
- $102 : \lim_{\lambda_0 \to 0} \lim_{\lambda_2 \to 0}$
- $210 : \lim_{\lambda_1 \to 0} \lim_{\lambda_0 \to 0}$

Example: Evaluating degrees of freedom

Recall

$$\lambda_0 + \lambda_1 + \lambda_2 = 1, \qquad \psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}.$$

Evaluating degrees of freedom

$$012: \lim_{\lambda_1\to 0}\lim_{\lambda_2\to 0}\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_1\to 0}\frac{\lambda_0\lambda_1}{\lambda_1} = \lim_{\lambda_0\to 1}\lambda_0 = 1,$$

$$021: \lim_{\lambda_2\to 0}\lim_{\lambda_1\to 0}\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_2\to 0}\frac{0}{\lambda_2} = 0,$$

$$120: \lim_{\lambda_2\to 0}\lim_{\lambda_0\to 0}\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_2\to 0}\frac{0}{1} = 0,$$

$$102: \lim_{\lambda_0\to 0}\lim_{\lambda_2\to 0}\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_0\to 0}\frac{\lambda_0\lambda_1}{\lambda_1} = 0,$$

$$201: \lim_{\lambda_0\to 0}\lim_{\lambda_1\to 0}\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_0\to 0}\frac{0}{\lambda_2} = 0,$$

$$210: \lim_{\lambda_1\to 0}\lim_{\lambda_0\to 0}\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_1\to 0}\frac{0}{1} = 0.$$

Motivation

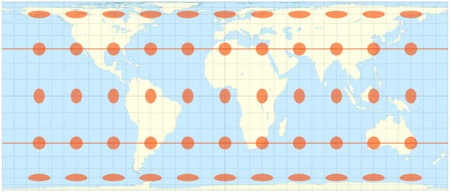
Motivating problem

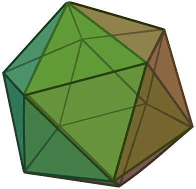
- Goal: construct intrinsic discretizations of tangent vector fields on smooth surfaces that are continuous across edges.
- Obstruction to using classical $\mathcal{P}_1$ elements: angle defect.

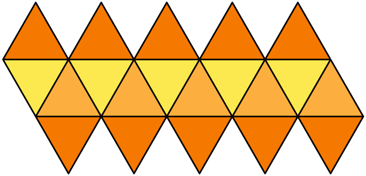
Remark about FEEC

- FEEC discretizations are intrinsic but only tangentially continuous across edges. Normal components are generally discontinuous.
- FEEC discretization suffices for Hodge Laplacian, but not for Bochner Laplacian.

Extrinsic vs. Intrinsic







Why compute intrinsically?

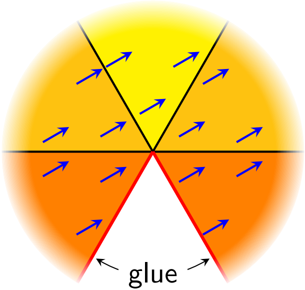
- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

Four images from Wikipedia

Angle defect obstruction to continuous elements

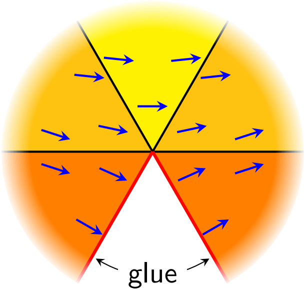
- Try to construct a tangent vector field on the icosahedron.
- What do we see when we zoom in on a vertex?

continuous elements



continuous on each triangle
discontinuous across red edge

blow-up elements



continuous across all edges
discontinuous on each triangle

Vector Laplacian eigenvalue problems

Hodge Laplacian

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

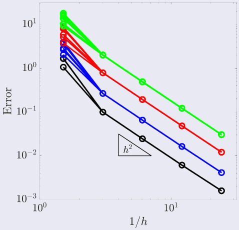
- Tangential continuity suffices.
- Standard FEEC works.

Bochner Laplacian

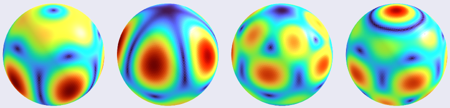
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.

Bochner Laplacian on sphere using blow-up elements



Eigenvalue error



Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

Blow-up Whitney Forms

This talk so far

- Lowest order blow-up elements in two dimensions, $b\mathcal{P}_1(T^2)$,
  - including vector fields with components in $b\mathcal{P}_1(T^2)$.

Recall: Whitney forms

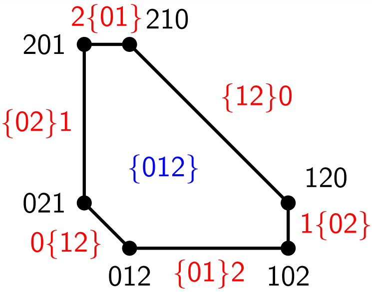
$$\mathcal{P}_1^-\Lambda^0(T^n) \xrightarrow{d} \mathcal{P}_1^-\Lambda^1(T^n) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_1^-\Lambda^n(T^n).$$

Blow-up Whitney forms

$$b\mathcal{P}_1^-\Lambda^0(T^n) \xrightarrow{d} b\mathcal{P}_1^-\Lambda^1(T^n) \xrightarrow{d} \cdots \xrightarrow{d} b\mathcal{P}_1^-\Lambda^n(T^n).$$

- Complex previously studied in (Brasselet, Goresky, MacPherson, 1991), called shadow forms.

Blow-up Whitney forms in 2D



Recall: one 0-form per vertex

$$\psi_{012} = \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2}$$

Nothing new for 2-forms

$$\psi_{\{012\}} = \varphi_{012}.$$

Similarly, one 1-form per edge

- For long edges, just the classical Whitney form:

$$\psi_{\{12\}0} = \varphi_{12} = \lambda_1\, d\lambda_2 - \lambda_2\, d\lambda_1.$$

- For short edges, something new:

$$\psi_{0\{12\}} = \frac{\lambda_0}{\lambda_1 + \lambda_2} \left(1 + \frac{1}{\lambda_1 + \lambda_2}\right) \varphi_{12}.$$

A surprising connection

Arrival times of Poisson processes

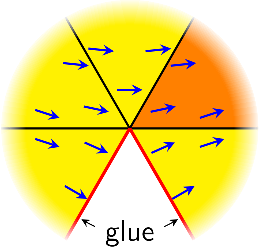
- Three radiation sources with rates λ_0, λ_1, and λ_2, sum 1.
- Let t_0, t_1, t_2 be the times when the respective radiation sources produce their first particle.
- $\psi_{012} = \lambda_0 \frac{\lambda_1}{\lambda_1+\lambda_2}$ is the probability that $t_0 \le t_1 \le t_2$.

What about one-forms?

- Let radiation source A have rate λ_0 and radiation source B have rate $\lambda_1 + \lambda_2$.
- Let t_A be the time when source A produces its first particle, and let t_B be the time when source B produces its second particle.
- Let $p_{0\{12\}}$ be the probability that $t_A \le t_B$. Then

$$\psi_{0\{12\}} = p_{0\{12\}} \frac{\varphi_0}{\lambda_0} \frac{\varphi_{12}}{(\lambda_1 + \lambda_2)^2}.$$

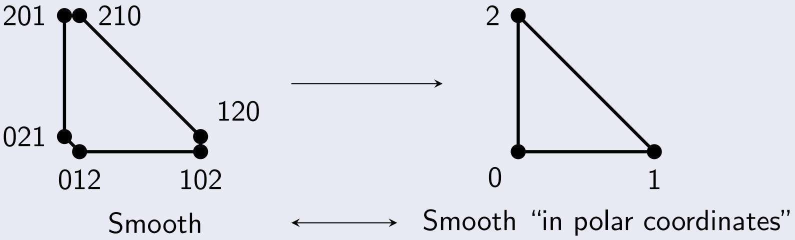
Blowing up



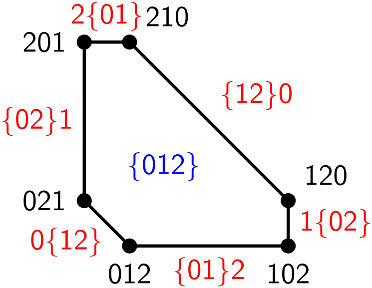
- Even on an individual triangle, the vector field is not continuous at the origin.
- But it is "continuous in polar coordinates," i.e. in r and θ.

Blowing up manifolds with corners (Melrose, 1996)

- formalizes continuity/smoothness "in polar coordinates"



Poisson process understanding of blowing up



Radiation rates

- Recall, three radiation sources with rates λ_0, λ_1, λ_2.
- Normalize time so total rate is 1.

What if $\lambda_0 \gg \lambda_1, \lambda_2$?

- Then, to floating point precision, $\lambda_0 = \lambda_0 + \lambda_1 + \lambda_2 = 1$.
- But then $\lambda_1 = \lambda_2 = 0$, so, classically, we cannot compare the rates of radiation sources 1 and 2.
- In the blow up, $\lambda_0 = 1$ along entire edge 0{12}, which is parametrized by $\frac{\lambda_1}{\lambda_1+\lambda_2}$.
- We can record $\lambda_0 : \lambda_1 : \lambda_2 = 1 : 0 : 0$ and $\lambda_1 : \lambda_2 = 3 : 5$.

Future directions

Higher order blow-up FEEC

- Higher-order blow-up scalar fields $b\mathcal{P}_r\Lambda^0(T^n)$ in our preprint.
- For general k-forms, in progress, joint with Michael Manta.

Analysis issues

- For blow-up scalar fields f in 2D, we have $\nabla f \in L^p$ for $p < 2$ but $\nabla f \notin L^2$, so $f \in W^{1,p}$ but $f \notin H^1$.
- Consequence: weak Bochner eigenvalue problem $\int \langle \nabla v, \nabla w \rangle \, dA = \lambda \int \langle v, w \rangle \, dA$ has infinite left-hand side.
- Workaround: Excise small nbhd of vertices. Works, but why?
- Note: Blow-up or not, tangent vector fields can't be in H^1.
  - 2nd derivative of vector fields yields curvature (angle defect).
  - Delta functions at vertices are not in H^{-1}.

Vector-valued or tensor-valued blow-up FEEC

- Vectors or tensors with components in $b\mathcal{P}_r^-\Lambda^k(T^n)$.

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik
Blow-up Whitney forms, shadow forms, and Poisson processes.
https://arxiv.org/abs/2402.03198, 2024.

J. P. Brasselet, M. Goresky, and R. MacPherson.
Simplicial differential forms with poles.
Amer. J. Math., 113(6):1019–1052, 1991.

R. B. Melrose.
Differential analysis on manifolds with corners.
https://math.mit.edu/~rbm/book.html, 1996.

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