Blow-up Finite Elements

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Florida Institute of Technology

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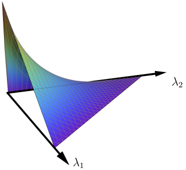
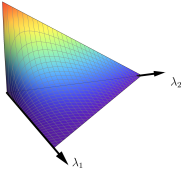
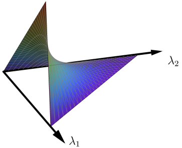
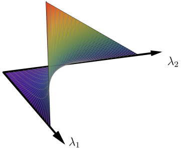
New finite element space



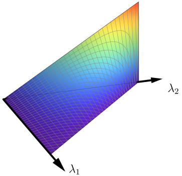


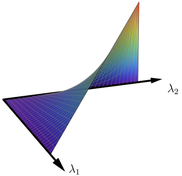
$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$





$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

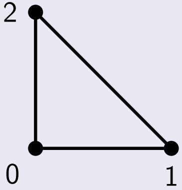




$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

Degrees of freedom

Classical $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
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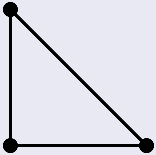

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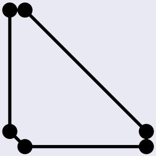
Blow-up $b\mathcal{P}_1$



- $012 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $120 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_0 \to 0}$
- $201 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $021 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $102 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $210 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_0 \to 0}$

Example: Evaluating degrees of freedom

Recall

$$\lambda_0 + \lambda_1 + \lambda_2 = 1, \qquad \psi_{012} = \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2}.$$

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Evaluating degrees of freedom

$$012: \lim_{\lambda_1 \to 0} \lim_{\lambda_2 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_1 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1} = \lim_{\lambda_0 \to 1} \lambda_0 = 1,$$

$$021: \lim_{\lambda_2 \to 0} \lim_{\lambda_1 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_2 \to 0} \frac{0}{\lambda_2} = 0,$$

$$120: \lim_{\lambda_2 \to 0} \lim_{\lambda_0 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_2 \to 0} \frac{0}{1} = 0,$$

$$102: \lim_{\lambda_0 \to 0} \lim_{\lambda_2 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_0 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1} = 0,$$

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$$210: \lim_{\lambda_1 \to 0} \lim_{\lambda_0 \to 0} \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2} = \lim_{\lambda_1 \to 0} \frac{0}{1} = 0.$$

Motivation

Motivating problem

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- Goal: construct intrinsic discretizations of tangent vector fields on smooth surfaces that are continuous across edges.

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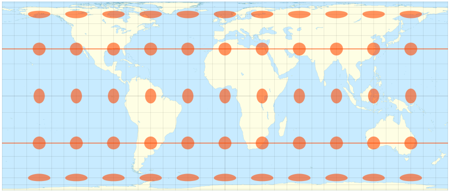
Remark about FEEC

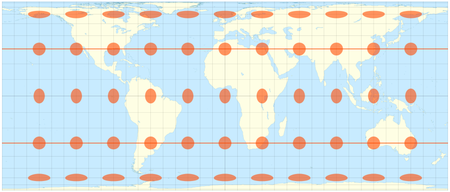
- FEEC discretizations are intrinsic but only tangentially continuous across edges. Normal components are generally discontinuous.
- FEEC discretization suffices for Hodge Laplacian, but not for Bochner Laplacian.

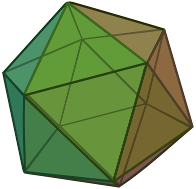
Extrinsic vs. Intrinsic

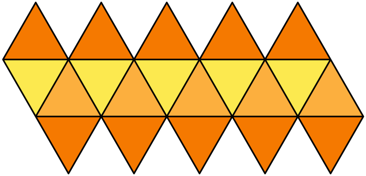


Four images from Wikipedia

Extrinsic vs. Intrinsic







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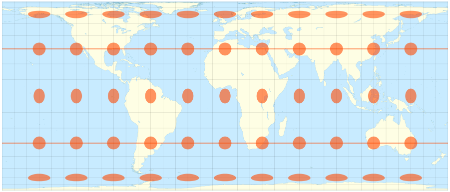

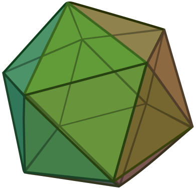

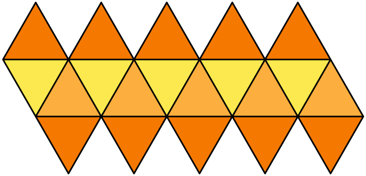

Why compute intrinsically?

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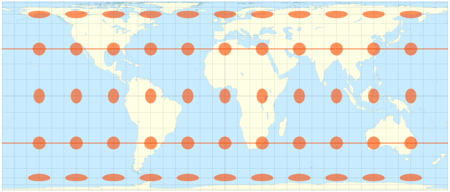

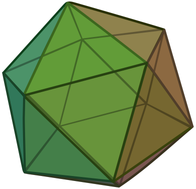

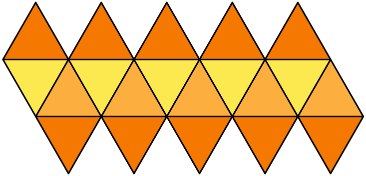

Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.

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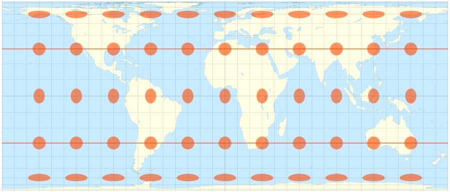

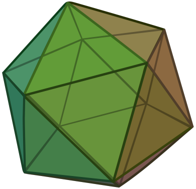

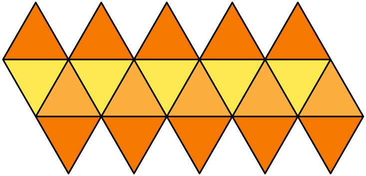

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- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

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Angle defect obstruction to continuous elements

- Try to construct a tangent vector field on the icosahedron.

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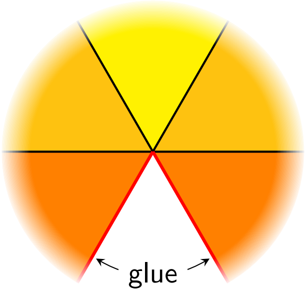

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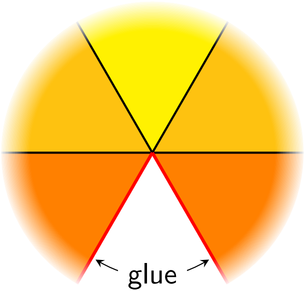

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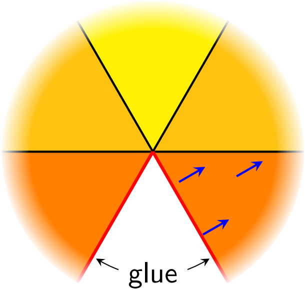

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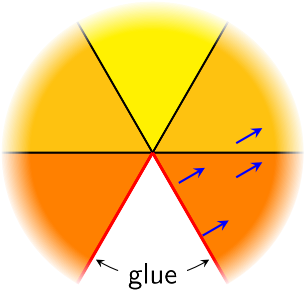

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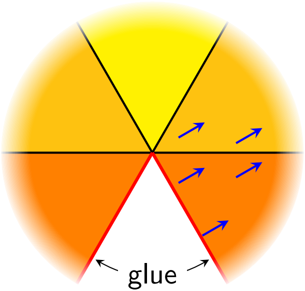

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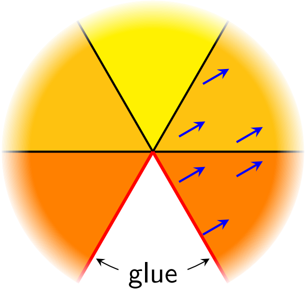

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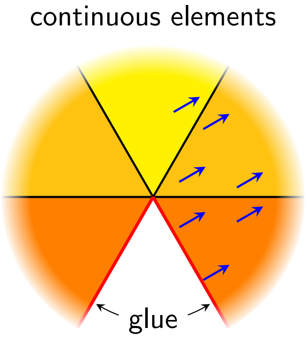

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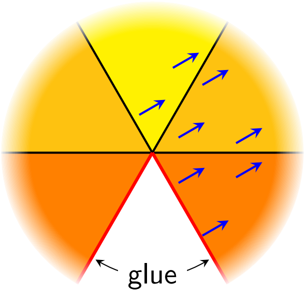

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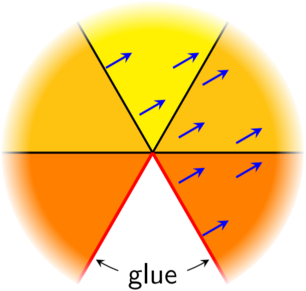

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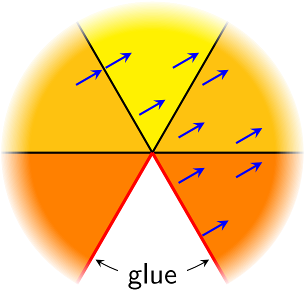

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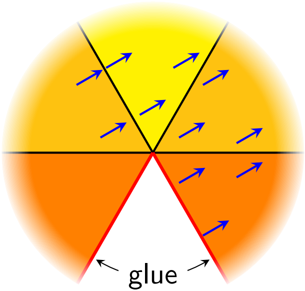

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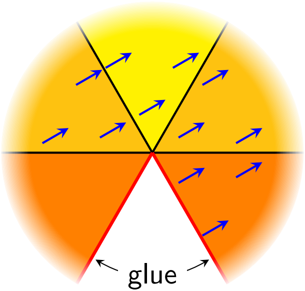

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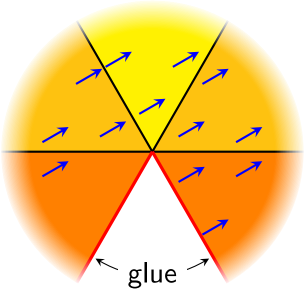

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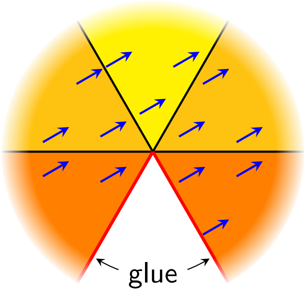

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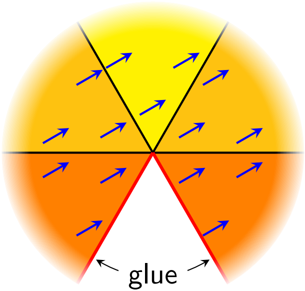

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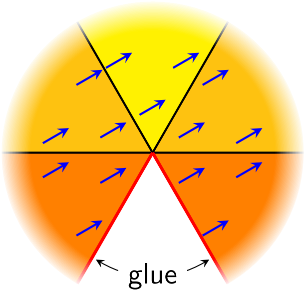

continuous on each triangle

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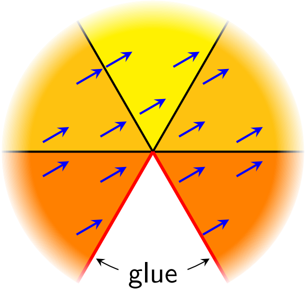

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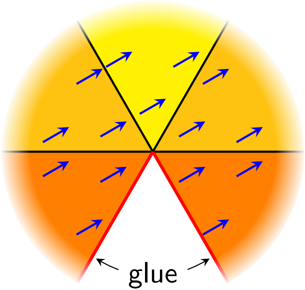
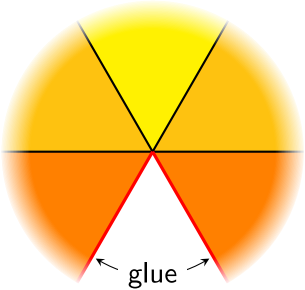


blow-up elements



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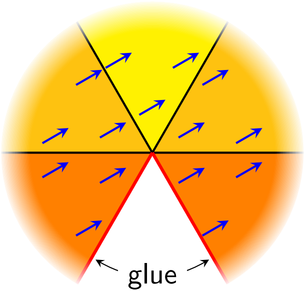

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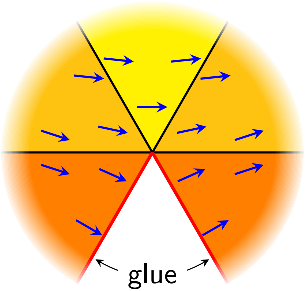

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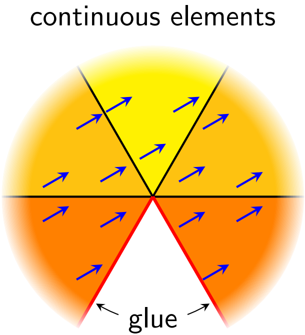

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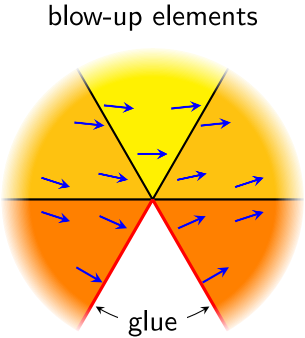

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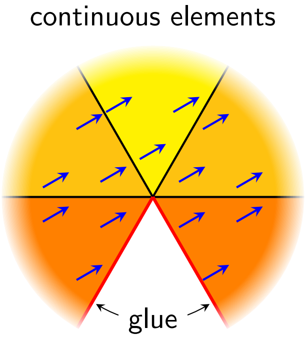

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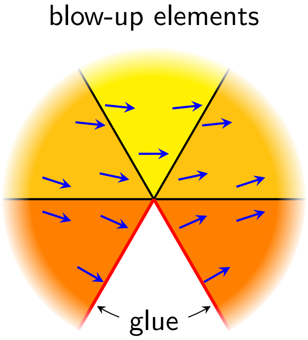

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Vector Laplacian eigenvalue problems

Hodge Laplacian

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

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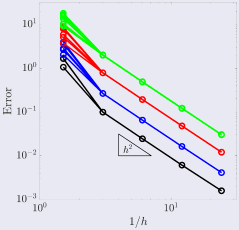
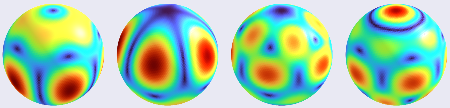
Bochner Laplacian on sphere using blow-up elements



Eigenvalue error



Eigenfield magnitude
$(\lambda = 11, 11, 19, 19)$

Blow-up Whitney Forms

This talk so far

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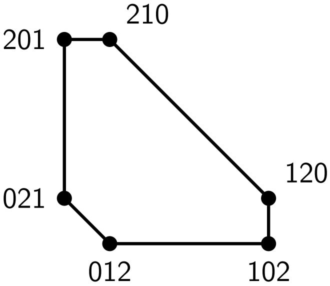
- Complex previously studied in (Brasselet, Goresky, MacPherson, 1991), called shadow forms.

Blow-up Whitney forms in 2D



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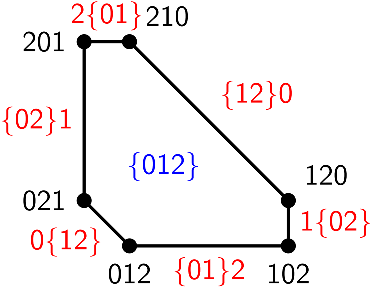

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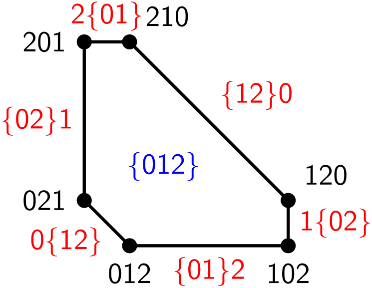

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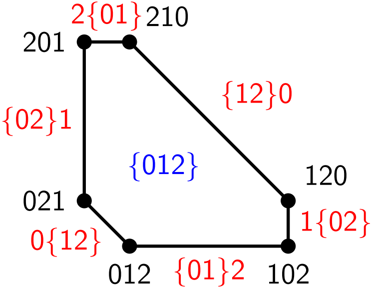

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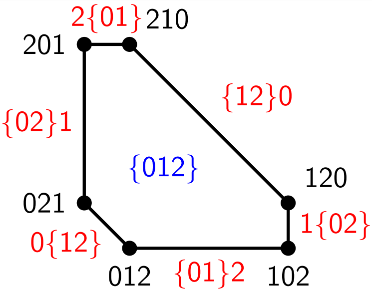

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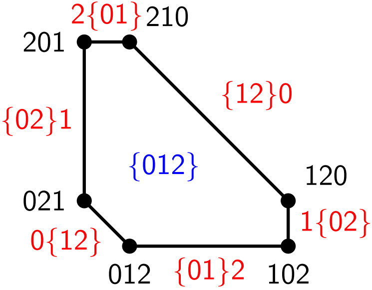

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Nothing new for 2-forms

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A surprising connection

Arrival times of Poisson processes

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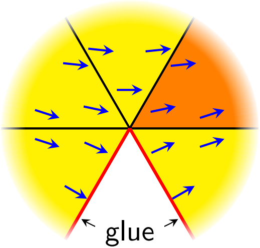
$$\psi_{0\{12\}} = p_{0\{12\}} \frac{\varphi_0}{\lambda_0} \frac{\varphi_{12}}{(\lambda_1 + \lambda_2)^2}.$$

Blowing up



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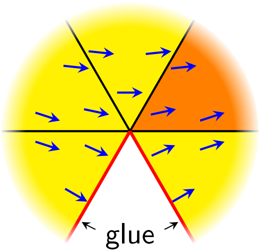

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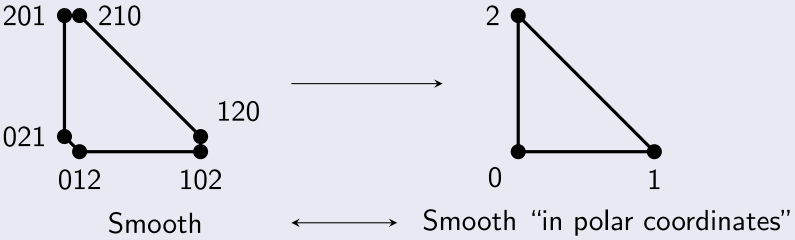
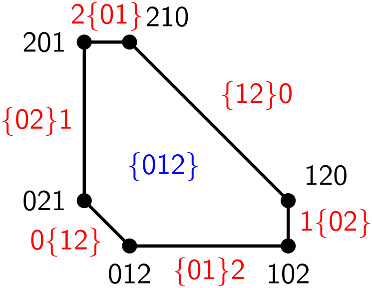
Blowing up manifolds with corners (Melrose, 1996)

- formalizes continuity/smoothness "in polar coordinates"



Poisson process understanding of blowing up



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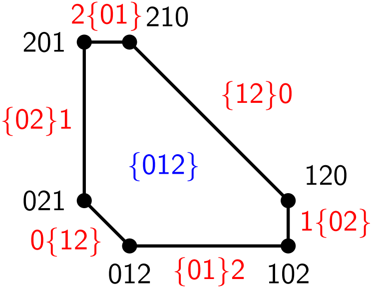

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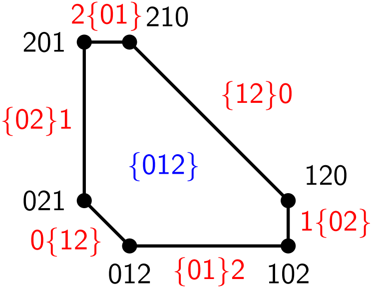

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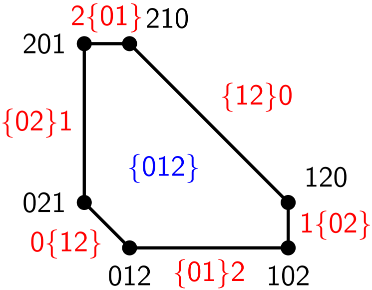

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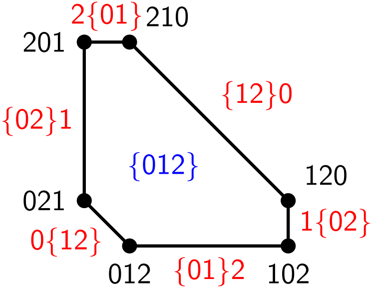

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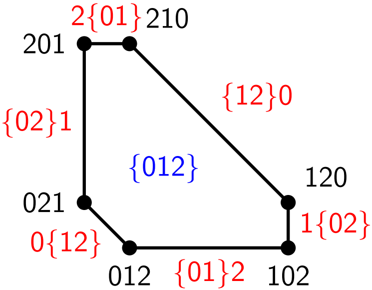

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- We can record $\lambda_0 : \lambda_1 : \lambda_2 = 1 : 0 : 0$ and $\lambda_1 : \lambda_2 = 3 : 5$.

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Vector-valued or tensor-valued blow-up FEEC

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Thank you

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