Finite element spaces for tensor fields with specified interelement continuity conditions

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Tangential and normal continuity of vector fields

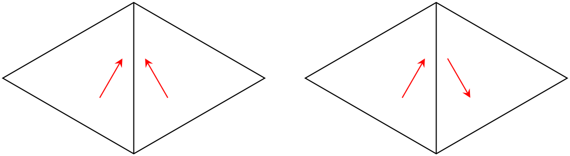


Figure: Tangential continuity (left) vs. normal continuity (right)

Tangential continuity

- Well-defined line integrals.
- In $H(\mathrm{curl})$.

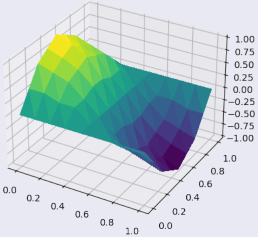
Normal continuity

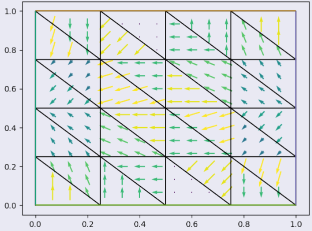
- Well-defined fluxes.
- In $H(\mathrm{div})$.

What's wrong with full continuity?

Finite element exterior calculus (FEEC) perspective: differential complexes

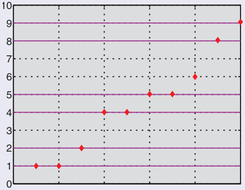
Gradients of scalar fields only have tangential continuity





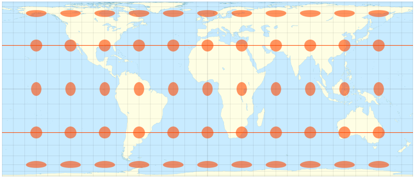
Spurious eigenvalues of the curl curl operator (AFW, 2010)

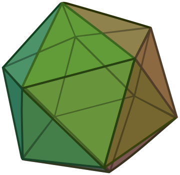
- Solve $\operatorname{curl}\operatorname{curl} u = \lambda u$, where u is a vector field on a square domain with appropriate boundary conditions.
- Using vector fields with full continuity yields false eigenvalue $\lambda = 6$.

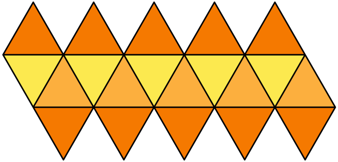


Extrinsic

Intrinsic







Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

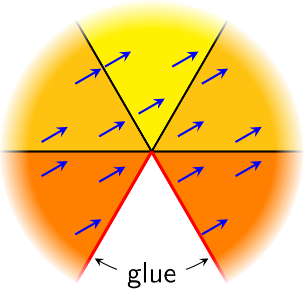
Four images from Wikipedia

What's wrong with full continuity?

Geometric perspective: Angle defect obstruction to continuous elements

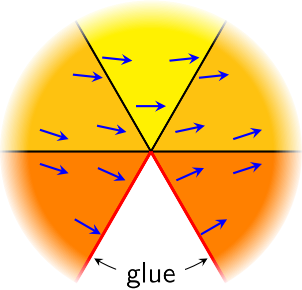
- Try to construct a tangent vector field on the icosahedron.
- What do we see when we zoom in on a vertex?

continuous elements



continuous on each triangle
discontinuous across red edge

blow-up elements



continuous across all edges
discontinuous at vertices

- See also later today: Alan Demlow (10am).

Differential forms corresponding to vector field $\langle M, N, P\rangle$

One-forms Λ^1

- $M\,dx + N\,dy + P\,dz$
- Restricted to the xy-plane $z = 0$:
  - $M\,dx + N\,dy$.
  - Tangential components.

Two-forms Λ^2

- $M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$.
- Restricted to the xy-plane $z = 0$:
  - $P\,dx \wedge dy$.
  - Normal component.

Continuity conditions

- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are $(n-1)$-forms.

Extending FEEC to matrices and tensors

Continuity conditions for 2-tensors (matrix fields)

- tangential–tangential
- normal–normal
- normal–tangential

Applications

- Strain/stress tensors
  - Elasticity (objects deforming under stress)
  - Fluid mechanics (Stokes equations)
- Numerical geometry/relativity
  - Riemannian/Minkowski metric
  - Curvature tensor
- See also later today: Francis Aznaran (9:30am), Alan Demlow (10am), Qingguo Hong (2pm), Bowen Shi (2:30pm).

Double forms

Vector fields ($\mathbb{R}^3$)

- Vector fields with tangential continuity are one-forms Λ^1.
- Vector fields with normal continuity are two-forms Λ^2.

Matrix fields ($\mathbb{R}^3 \otimes \mathbb{R}^3$)

- Matrix fields with tangential–tangential continuity are $(1,1)$-forms $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$.
- Matrix fields with normal–tangential continuity are $(2,1)$-forms $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$.
- Matrix fields with normal–normal continuity are $(2,2)$-forms $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$.

More on double forms later today

- Evan Gawlik (8:30am).
- Anil Hirani (9am).

Affine-invariance vs. metric-dependence

Affine-invariant (metric-independent) finite element spaces

- FEEC differential forms Λ^k and their continuity conditions are defined without reference to a Riemannian metric.
- Same for double forms $\Lambda^{p,q}$.
- Angle defect cannot pose a problem since angle defect is not even defined without a Riemannian metric.
- In particular, for vector fields with tangential or normal continuity, FEEC works just as well on surface meshes as it does on the plane.

Metric-dependent finite element spaces

- Defining finite element spaces of vector fields with full continuity requires a Riemannian metric (even via differential form proxies).
- Behavior depends on whether angle defect is zero or not.

Affine-invariant subspaces of double forms

Theorem (Eigendecomposition of s^*s)

$$\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m, \qquad \max\{0, q-p\} \le m \le \min\{q, n-p\}.$$

Example

- $\Lambda^{1,1}_0$: Symmetric bilinear forms, $\varphi(X;Y) = \varphi(Y;X)$.
- $\Lambda^{1,1}_1$: Λ^2, antisymmetric bilinear forms, $\varphi(X;Y) = -\varphi(Y;X)$.

- $\Lambda^{2,1}_0$: spanned by $\alpha \otimes \beta$ such that $\alpha \wedge \beta = 0$.
  - Matrix proxy in 3D: trace-free matrices.
- $\Lambda^{2,1}_1$: Λ^3.
  - Matrix proxy in 3D: multiples of the identity matrix.

- $\Lambda^{2,2}_0$: Symmetric, satisfying the algebraic Bianchi identity.
  - Riemann curvature tensor.
- $\Lambda^{2,2}_1$: Antisymmetric, $\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$.
- $\Lambda^{2,2}_2$: Λ^4.

Finite element spaces

Theorem (—, Gawlik)

Apart from $\Lambda^{p,q}_q \cong \Lambda^{p+q}$ with constant coefficients, there is a finite element space for every natural space of double forms $\Lambda^{p,q}_m$ with polynomial coefficients of any degree (including zero).

Example (Constant coefficient spaces)

- $\Lambda^{1,1}_0$: symmetric matrices with tangential–tangential continuity (Regge, 1961).
    - Higher order: (Li, 2018).
- $\Lambda^{2,1}_0$ in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda^{2,2}_0$ in 3D: symmetric matrices with normal–normal continuity (Pechstein and Schöberl, 2011).
- $\Lambda^{2,2}_0$ (or $\Lambda^{n-2,n-2}_0$) in any dimension: finite elements for the Riemann curvature tensor.

Degrees of freedom for constant coefficient spaces

	d						
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	**1**	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	**2**	0	0	0	0
$\Lambda_0^{2,2}$	0	0	**1**	**2**	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	**3**	0	0	0
$\Lambda_0^{3,2}$	0	0	0	**3**	**5**	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	**4**	0	0
$\Lambda_0^{3,3}$	0	0	0	**1**	**5**	**5**	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	**6**	**9**	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	**5**	0

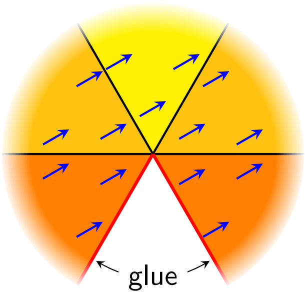
Table: Number of degrees of freedom for $\Lambda_m^{p,q}$ associated to a face of the triangulation of dimension d is $\frac{p-q+2m+1}{p+m+1}\binom{d+1}{q-m}\binom{q-m-1}{d-p-m}$.

Metric-dependent finite element spaces

Motivating problem

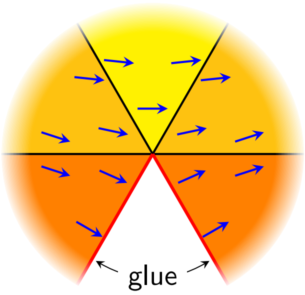
- Goal: construct intrinsic discretizations of tangent vector fields on smooth surfaces that are continuous across edges.
- Obstruction to using classical Lagrange $\mathcal{P}_1$ elements: angle defect.

continuous elements



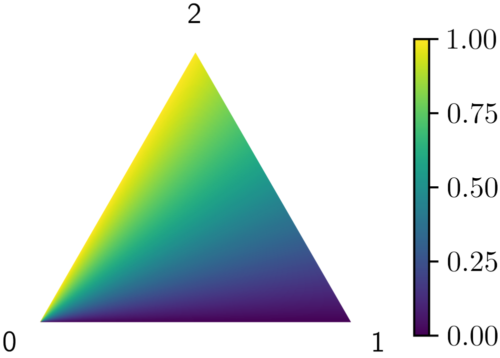
continuous on each triangle
discontinuous across red edge

blow-up elements



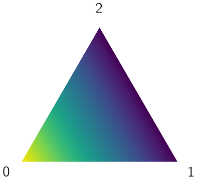
continuous across all edges
discontinuous at vertices

A simplicial analogue of the angular coordinate

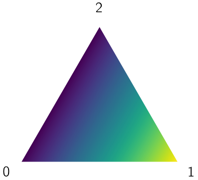


$$\frac{\lambda_2}{\lambda_1 + \lambda_2}$$

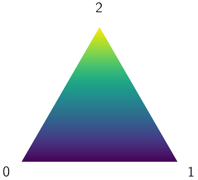
Lagrange $\mathcal{P}_1$ shape functions



λ_0

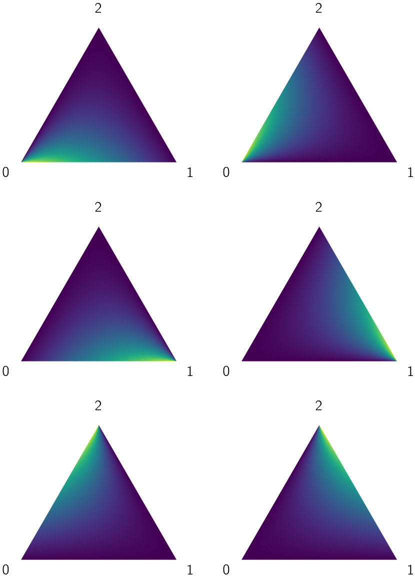


λ_1



λ_2

Blow-up $b\mathcal{P}_1$ shape functions



$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

New?

Shape function

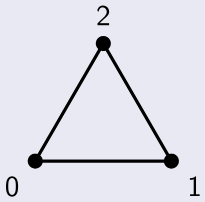
$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2} = \frac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_2}{\lambda_2}.$$

Earlier appearances

- Geometric invariants (Chen, 1957).
- Horse betting (Harville, 1973).
- Intersection homology (Brasselet, Goresky, MacPherson, 1991; Bendiffalah, 1995).

Degrees of freedom

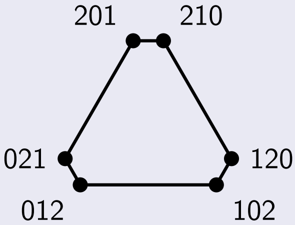
Classical Lagrange $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

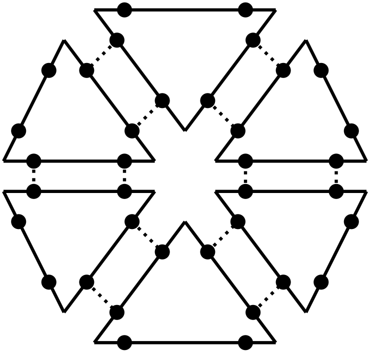
- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
- $2 : \lambda_2 = 1 \Leftrightarrow \lambda_0 = \lambda_1 = 0$

Blow-up $b\mathcal{P}_1$



- $012 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $120 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_0 \to 0}$
- $201 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $021 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $102 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $210 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_0 \to 0}$

Global spaces



Blow-up finite elements

- Scalar fields: we place a number at each dot.
- Vector fields: we place two numbers at each dot, for the tangential and normal components, respectively.
    - Enforce continuity for both components, yielding full continuity across edges.
- Matrix fields: At each dot, we record the tangential–tangential component, the tangential–normal component, etc.
    - Can impose conditions on the components such as symmetry, trace-free, etc.
    - Can enforce continuity for all components or just some of them.
- General tensor fields are analogous.

Vector Laplacian eigenvalue problems on surfaces

Hodge Laplacian

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

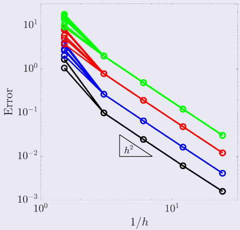
- Tangential continuity across edges suffices.
- Standard FEEC works.
- L^2 pairing suffices.

Bochner Laplacian

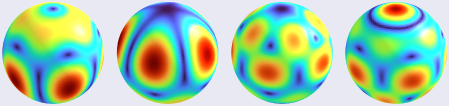
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

Bochner Laplacian on sphere using blow-up elements



Eigenvalue error



Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

There's more

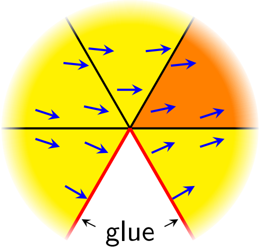
So far in this talk

- Lowest order blow-up elements in two dimensions, $b\mathcal{P}_1(T^2)$,
  - including tensor fields with components in $b\mathcal{P}_1(T^2)$.

Our paper

- Differential complex of blow-up Whitney forms in any dimension, $b\mathcal{P}_1^- \Lambda^k(T^n)$.
  - Shape functions previously studied in (Brasselet, Goresky, MacPherson, 1991), called shadow forms.
- Higher-order blow-up scalar fields $b\mathcal{P}_r(T^n)$.
- A surprising connection to arrival times of Poisson processes, yielding simpler computations.
  - Three radiation sources with rates λ_0, λ_1, and λ_2, sum 1.
  - Let t_0, t_1, t_2 be the times when the respective radiation sources produce their first particle.
  - $\frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2}$ is the probability that $t_0 \leq t_1 \leq t_2$.
- Degrees of freedom in terms of blow-up simplex.

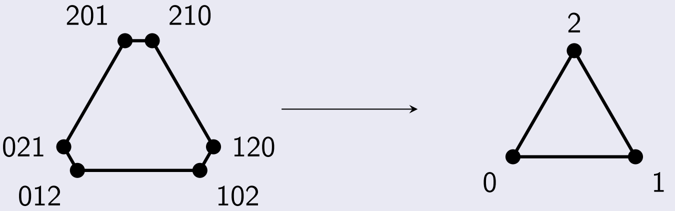
Blowing up



- Even on an individual triangle, the vector field is not continuous at the origin.
- But it is "continuous in polar coordinates," i.e. in r and θ.

Blowing up manifolds with corners (Melrose, 1996)

- formalizes continuity/smoothness "in polar coordinates"



Smooth $\longleftrightarrow$ Smooth "in polar coordinates"

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik
Finite element spaces of double forms.
https://arxiv.org/abs/2505.17243

Yakov Berchenko-Kogan
Duality in finite element exterior calculus and Hodge duality on the sphere.
Found. Comput. Math. 21(5):1153–1180, 2021.

Evan S. Gawlik and Anil N. Hirani
Sequences from sequences, sans coordinates.
In preparation.

Yakov Berchenko-Kogan and Evan S. Gawlik
Blow-up Whitney forms, shadow forms, and Poisson processes.
Results in Applied Mathematics, special issue on Hilbert complexes, Paper No. 100529, 2025.

J. P. Brasselet, M. Goresky, and R. MacPherson.
Simplicial differential forms with poles.
Amer. J. Math., 113(6):1019–1052, 1991.

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