Two approaches to finite element tensor calculus on surfaces

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Florida Institute of Technology
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May 21, 2026

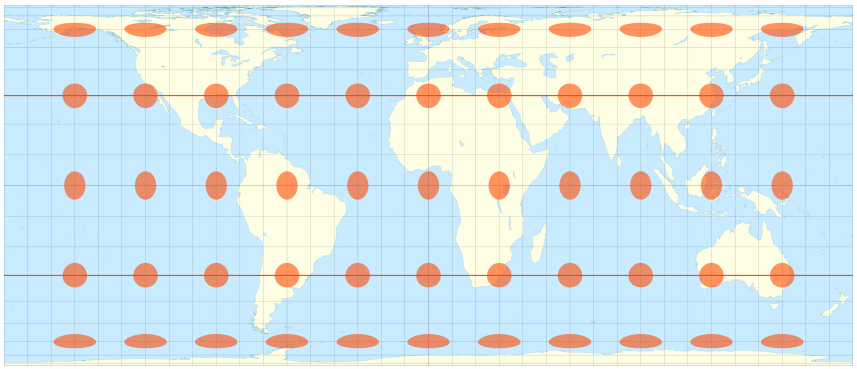
Section 1

Blow-up finite elements

Surfaces: extrinsic vs. intrinsic

Extrinsic

Intrinsic



Four images from Wikipedia

Surfaces: extrinsic vs. intrinsic

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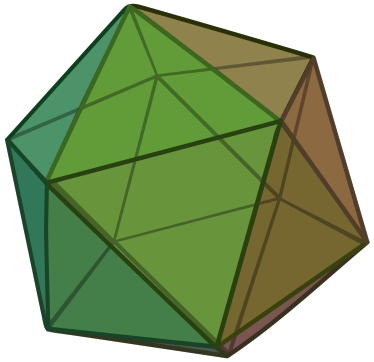

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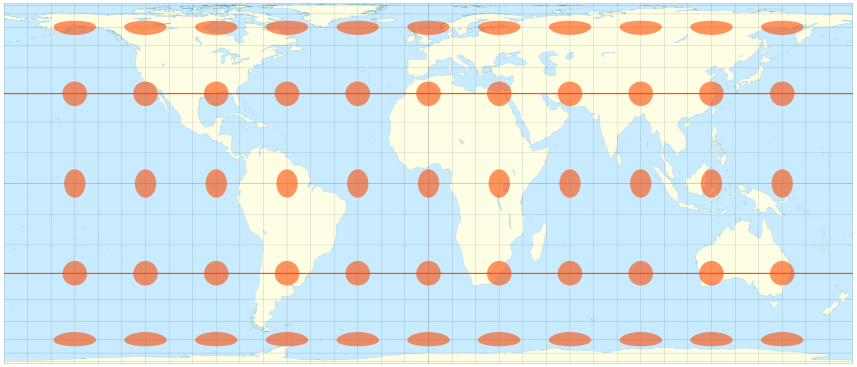

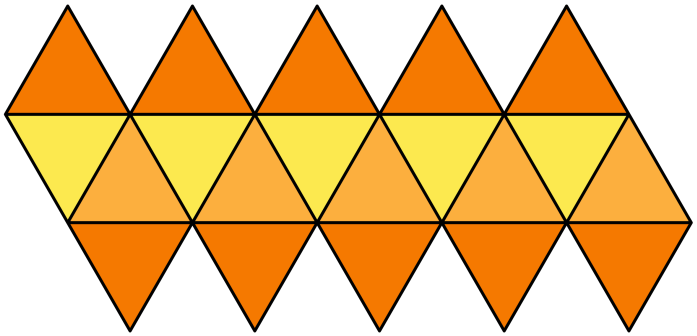

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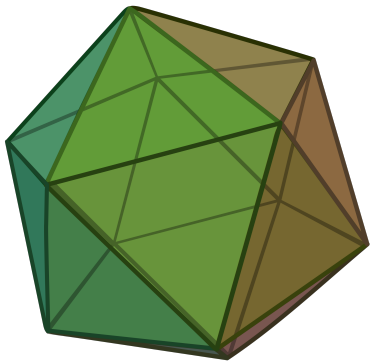

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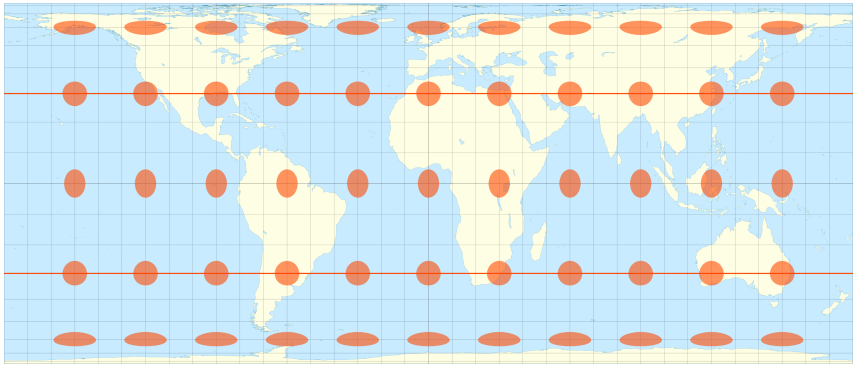

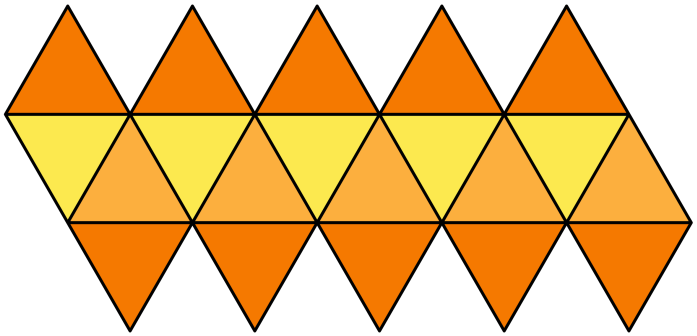

Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

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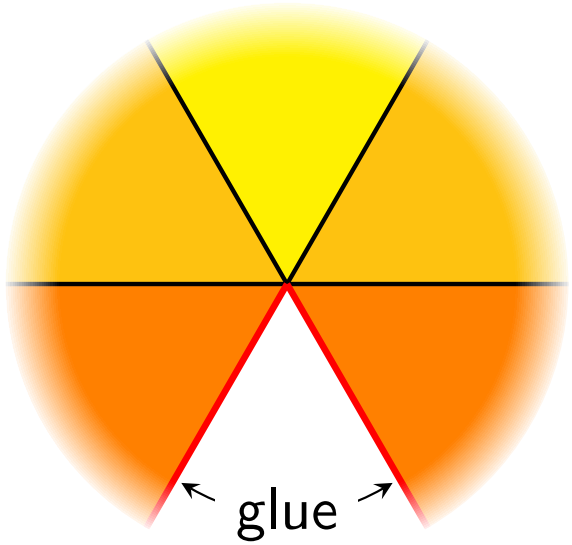

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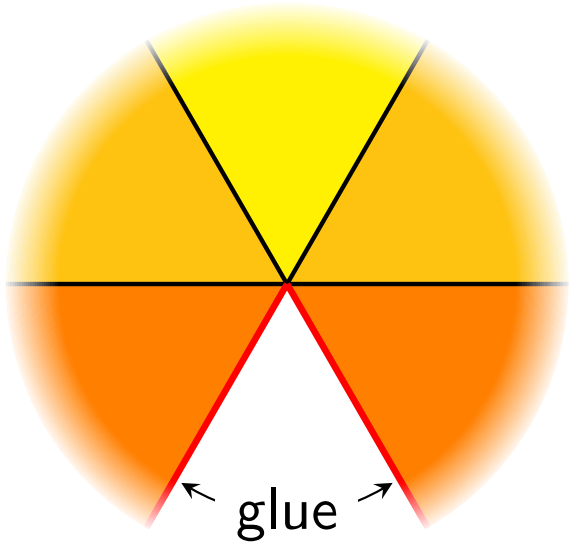

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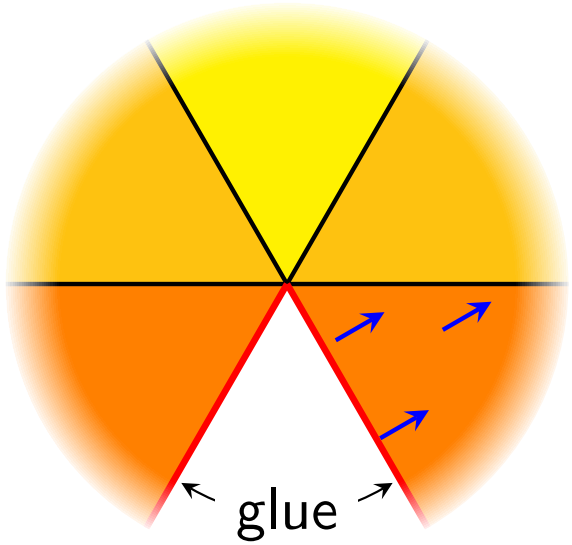

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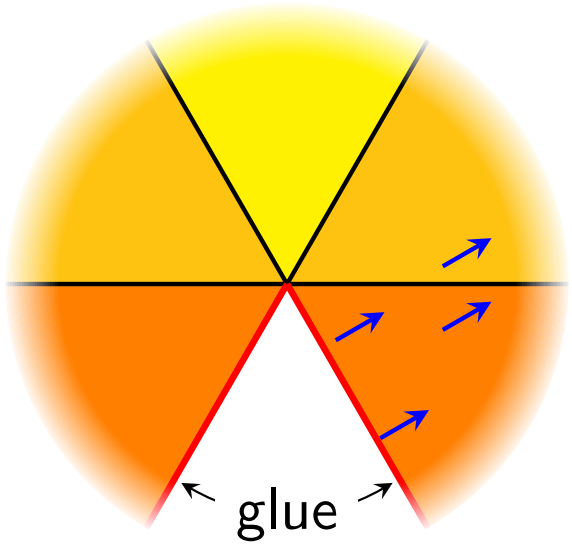

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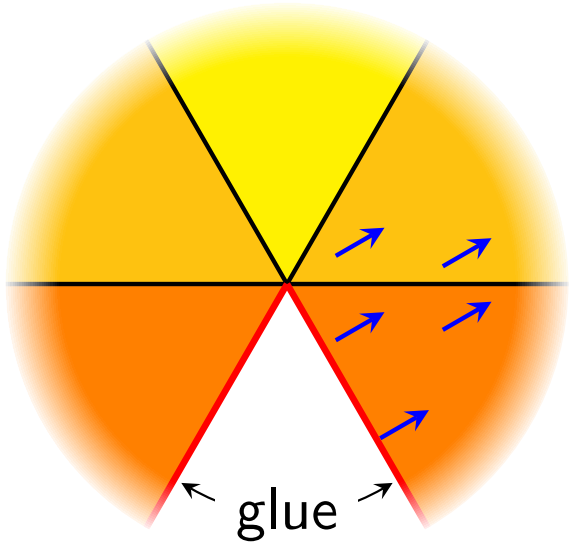

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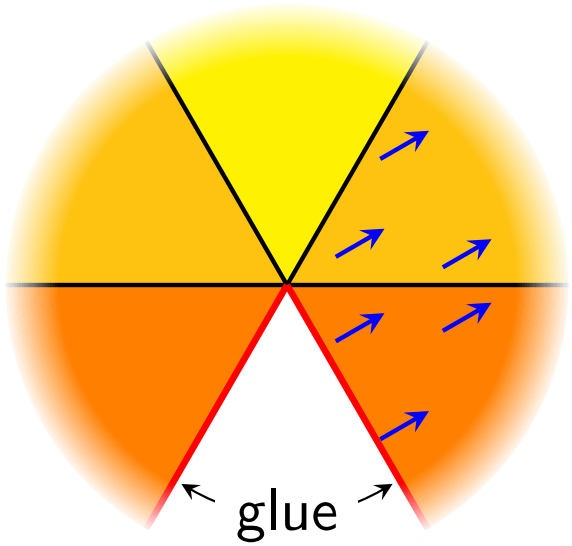

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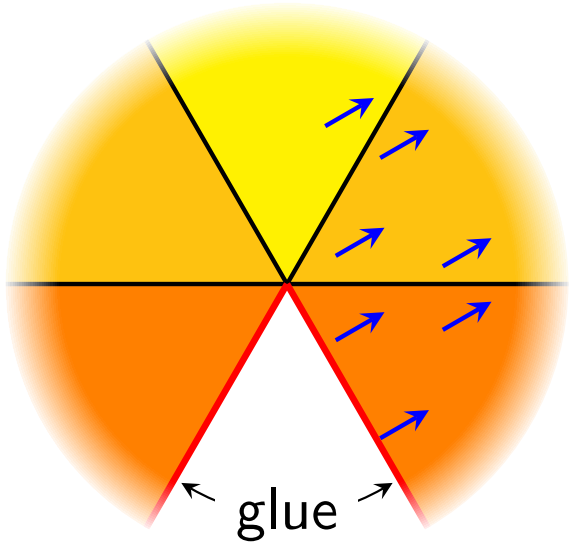

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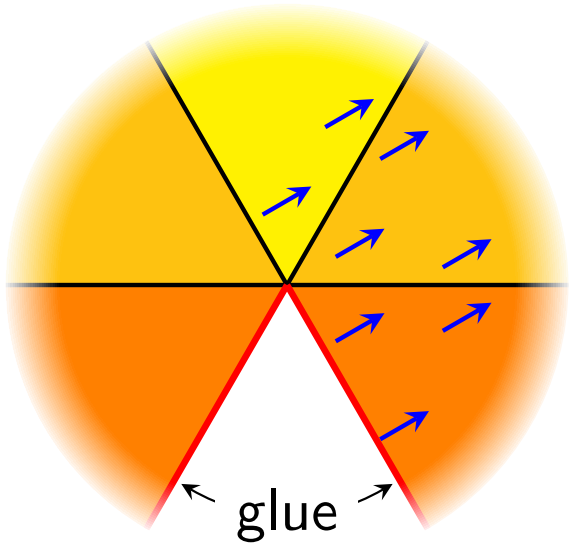

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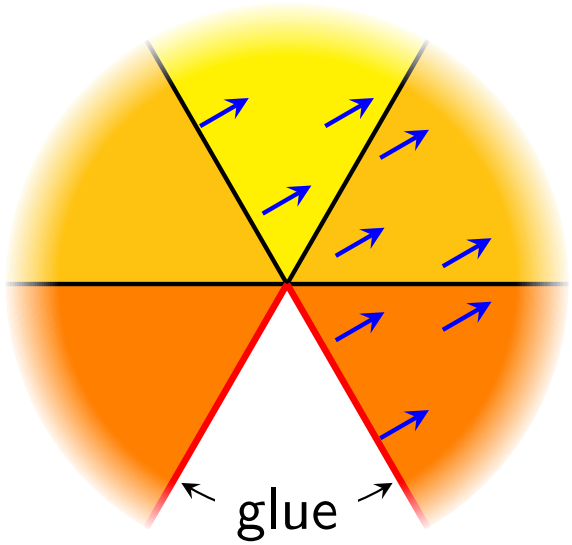

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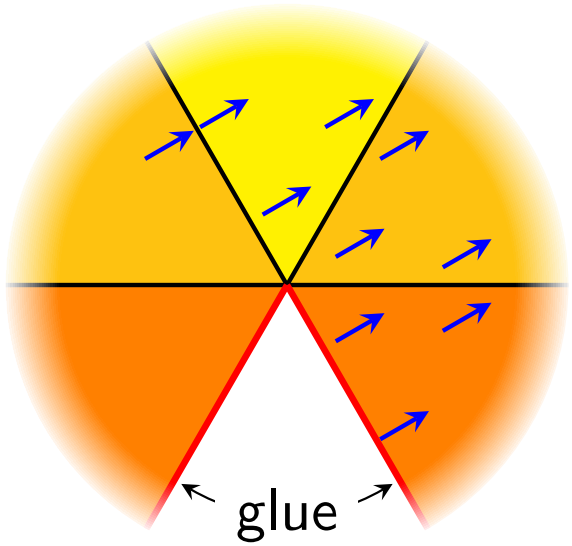

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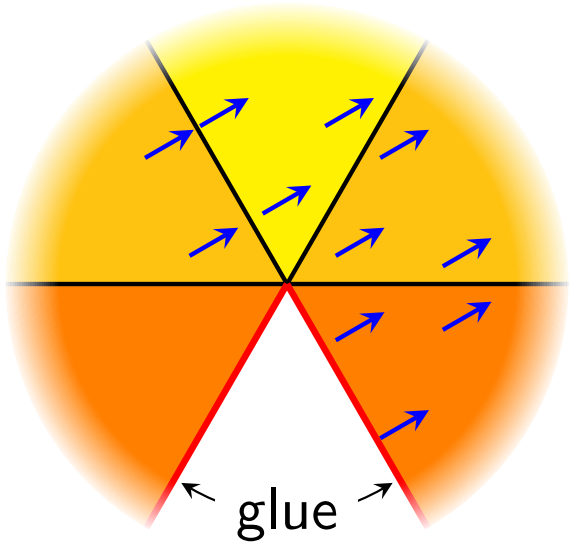

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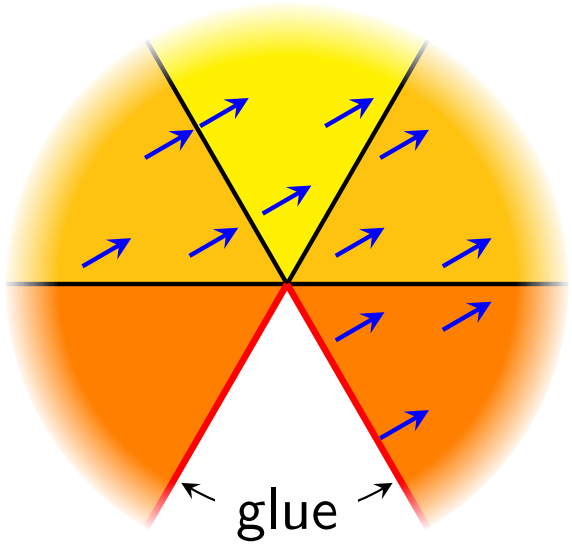

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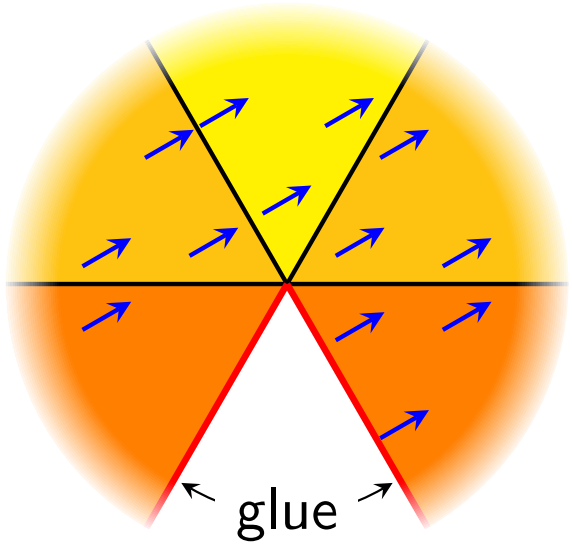

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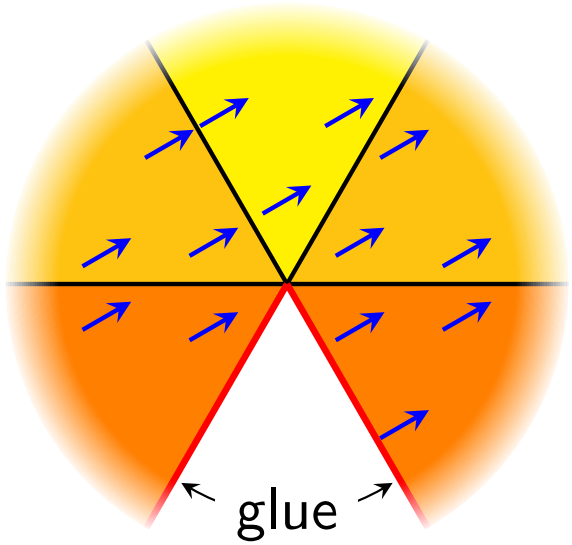

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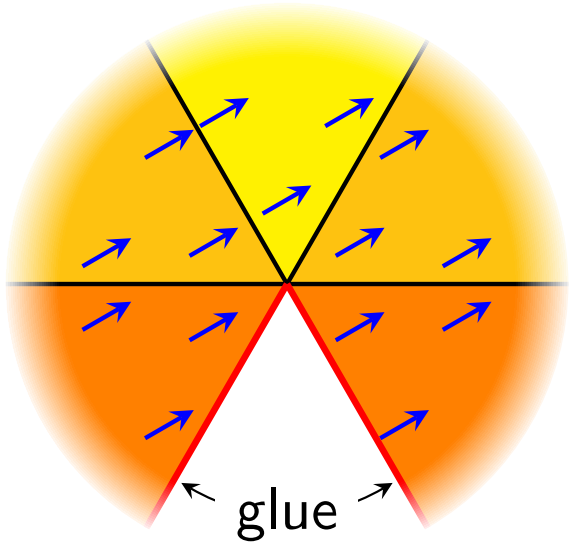

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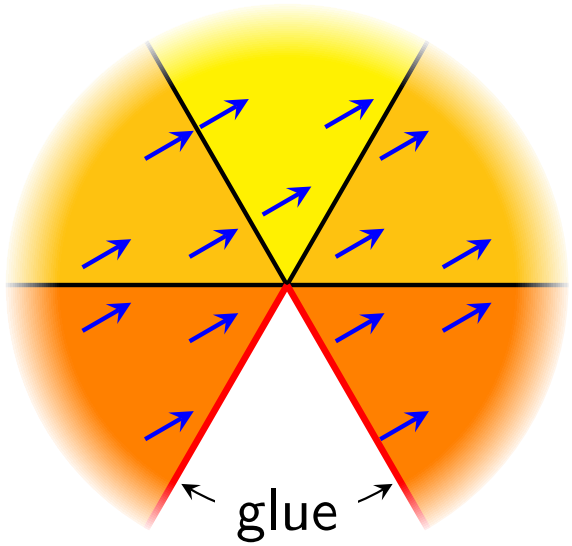

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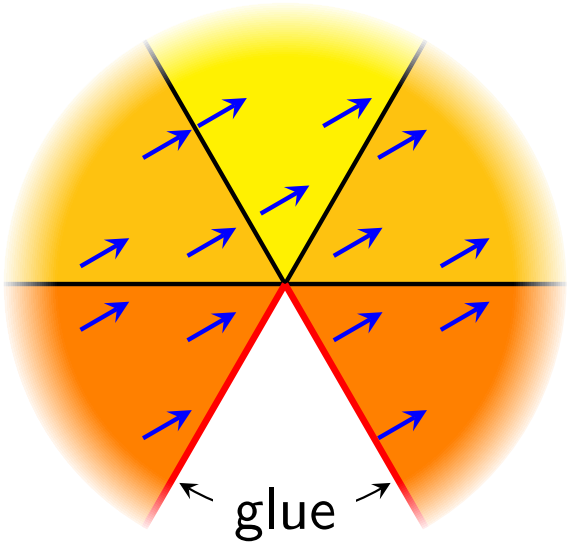

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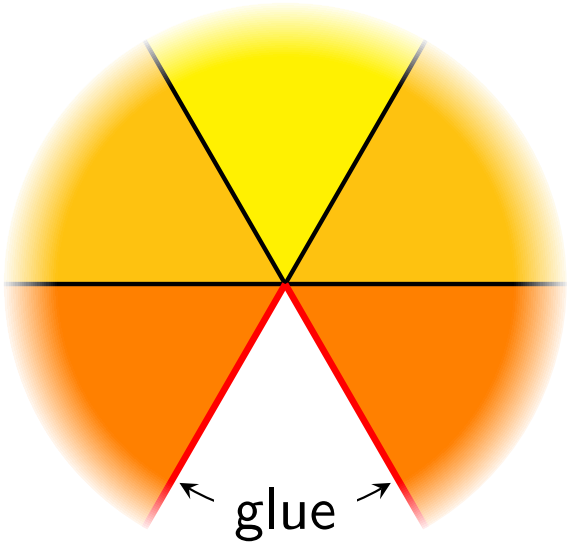

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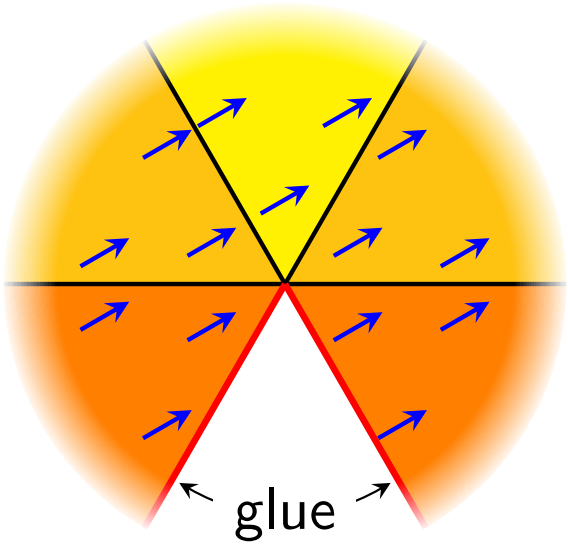

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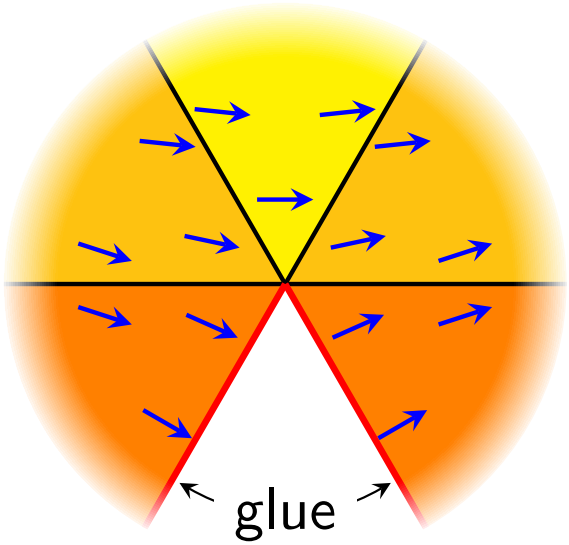

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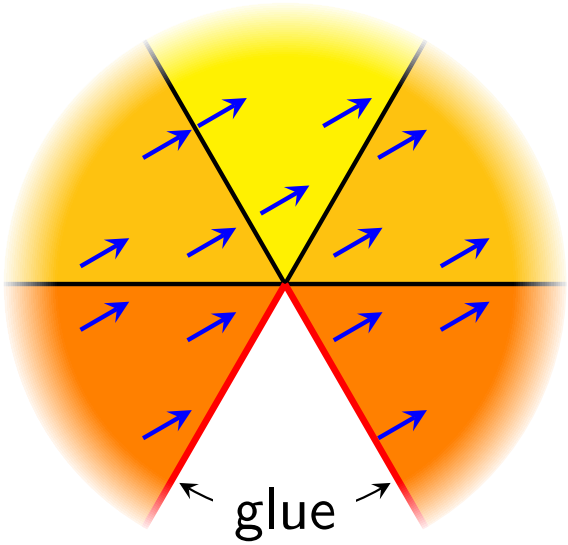

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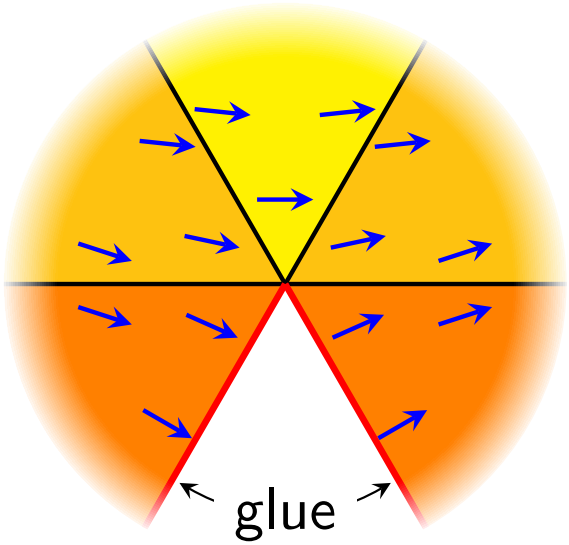

continuous across all edges

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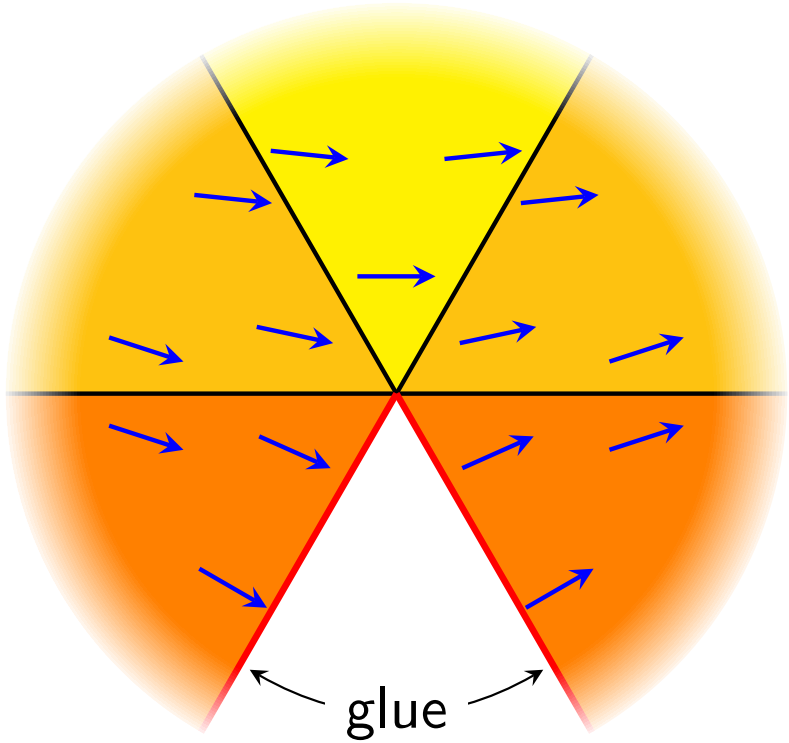
  - Discontinuities are *only* where the problem is (vertices with angle defect).

Break continuity at vertices

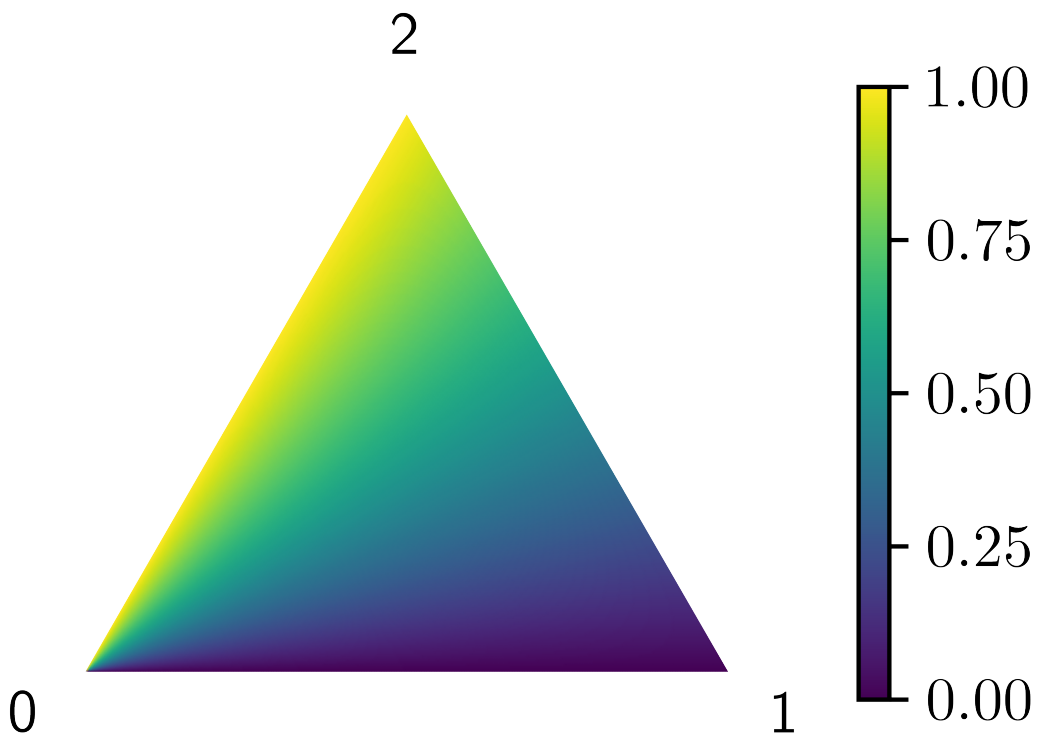
blow-up elements



continuous across all edges

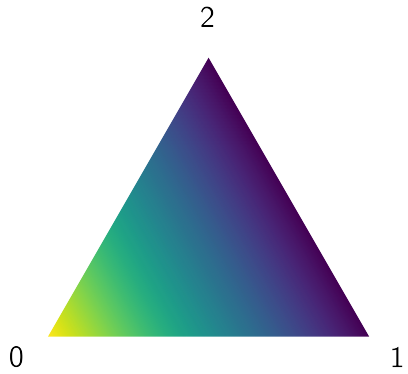
discontinuous at vertices

A simplicial analogue of the angular coordinate

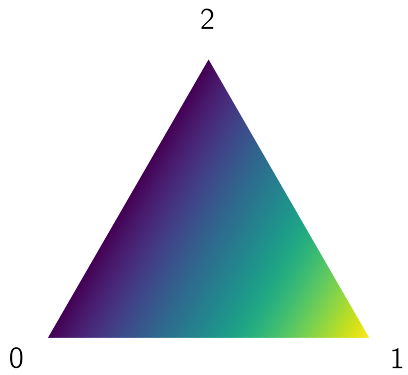


$$\frac{\lambda_2}{\lambda_1 + \lambda_2}$$

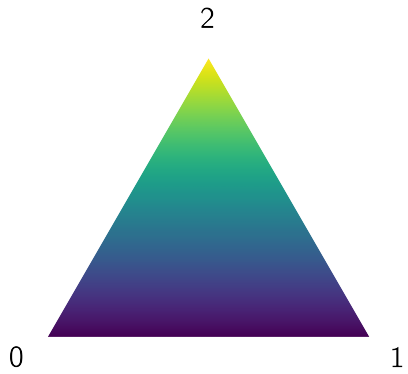
Lagrange $\mathcal{P}_1$ shape functions



λ_0

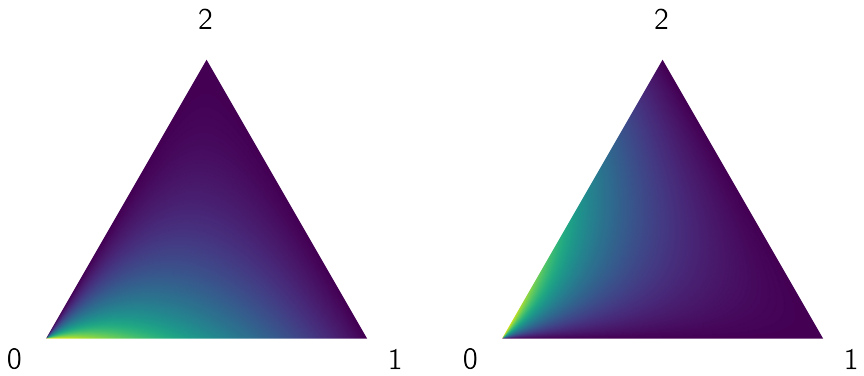


λ_1

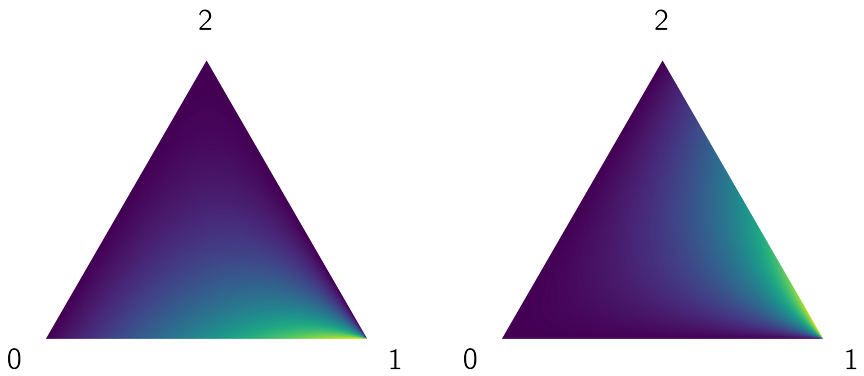


λ_2

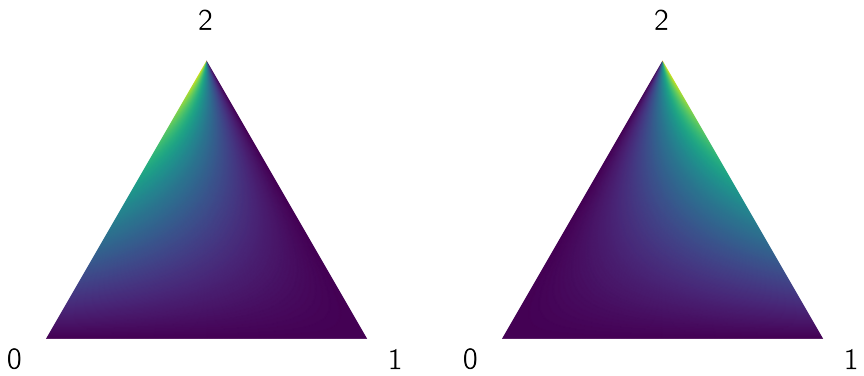
Blow-up $b\mathcal{P}_1$ shape functions



$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2+\lambda_1},$$



$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0+\lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2+\lambda_0},$$



$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0+\lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1+\lambda_0}.$$

New?

Shape function

$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{\lambda_2}{\lambda_2}.$$

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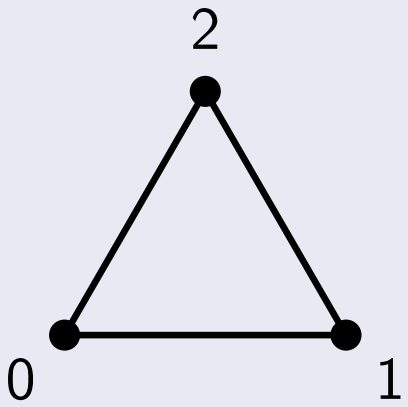
- Let t_0, t_1, t_2 be the times when the respective radiation sources produce their first particle.
- ψ_{012} is the probability that $t_0 \leq t_1 \leq t_2$.

Degrees of freedom

Classical Lagrange $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
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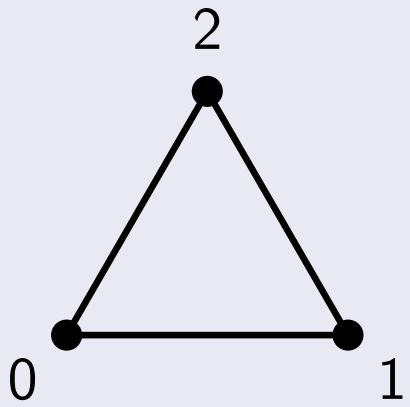

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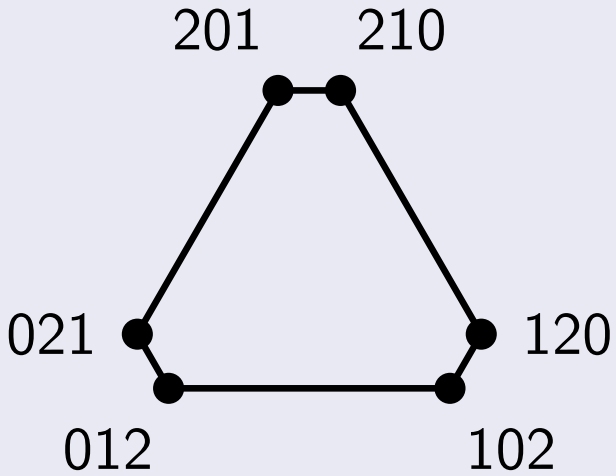
Blow-up $b\mathcal{P}_1$



- $012 : \lim_{\lambda_1 \to 0} \lim_{\lambda_2 \to 0}$
- $120 : \lim_{\lambda_2 \to 0} \lim_{\lambda_0 \to 0}$
- $201 : \lim_{\lambda_0 \to 0} \lim_{\lambda_1 \to 0}$
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- $102 : \lim_{\lambda_0 \to 0} \lim_{\lambda_2 \to 0}$
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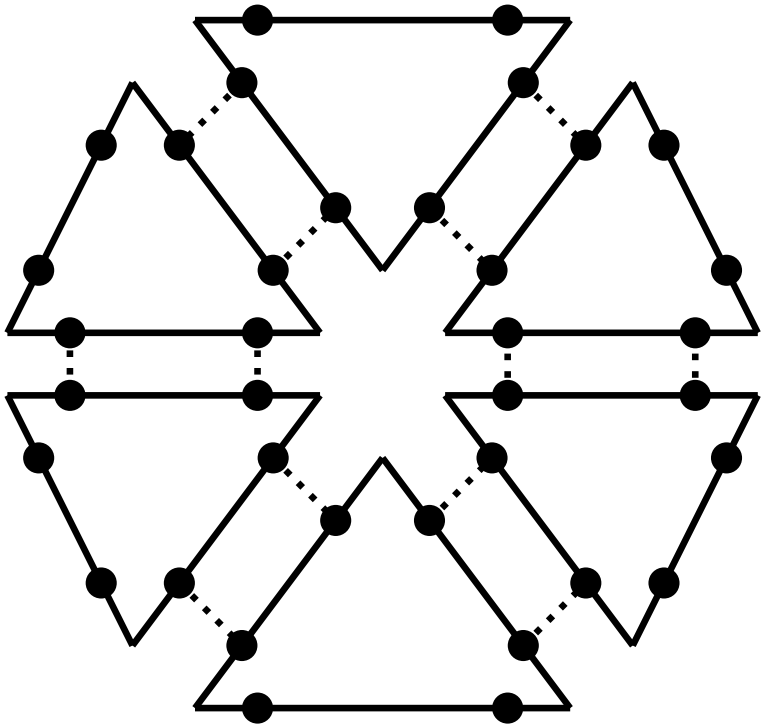
Global spaces



Blow-up finite elements

- Scalar fields: we place a number at each dot.

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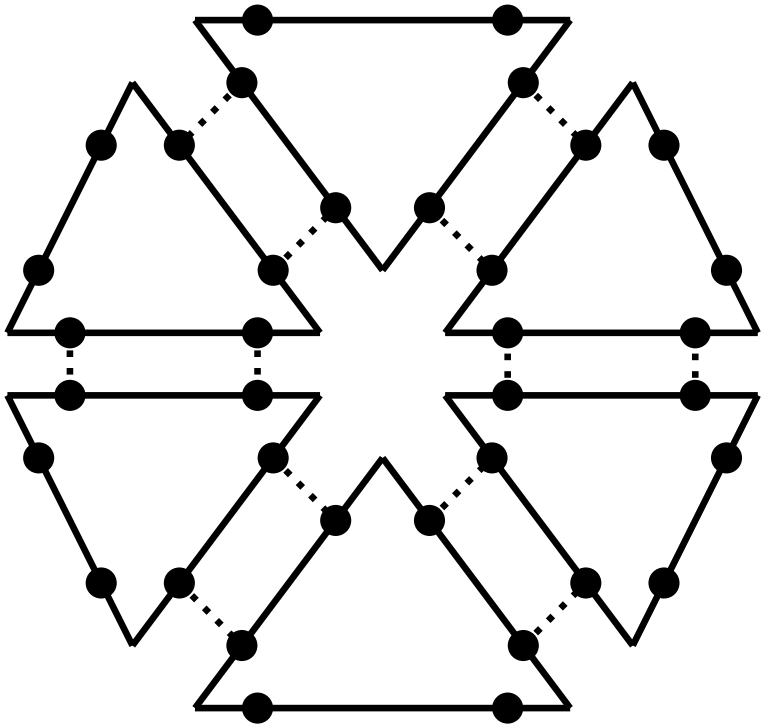

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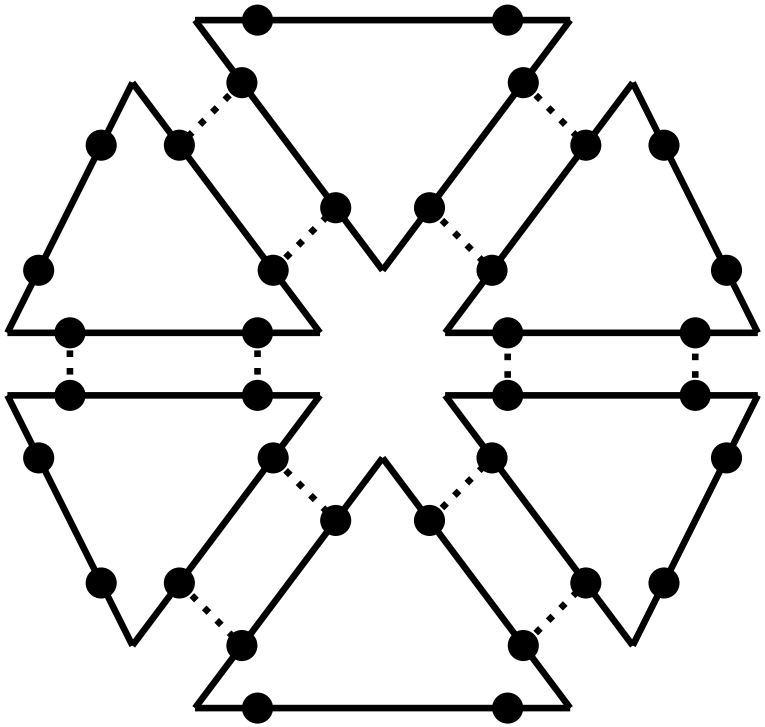

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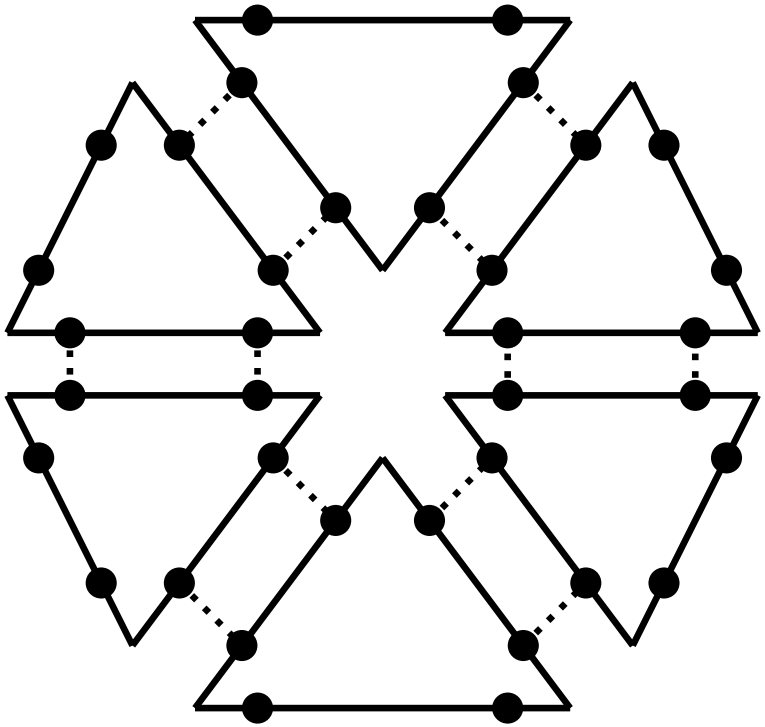

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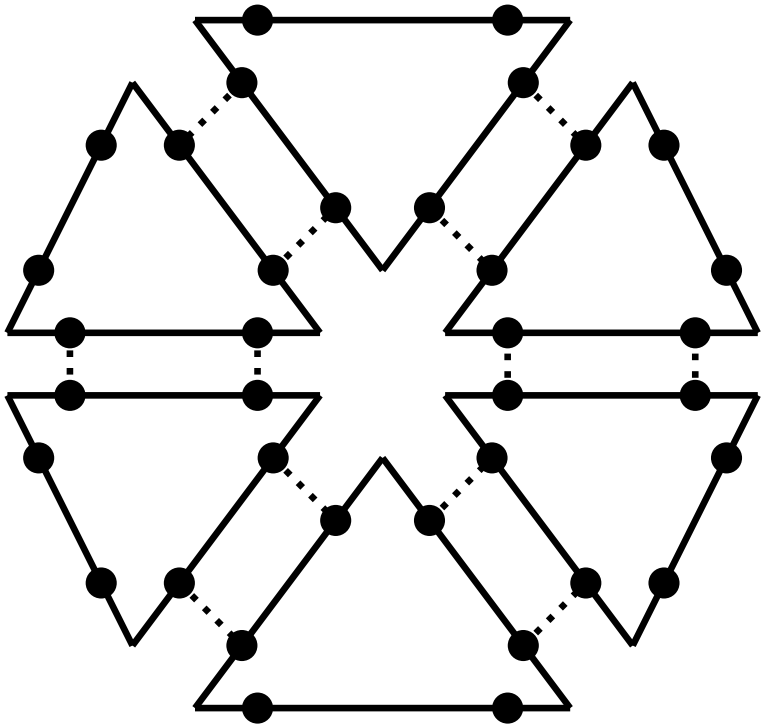

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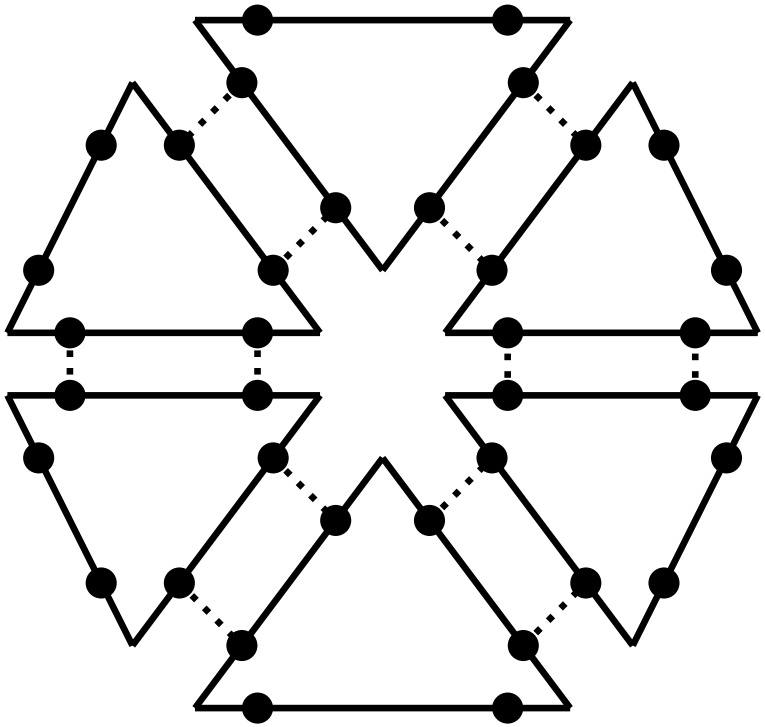

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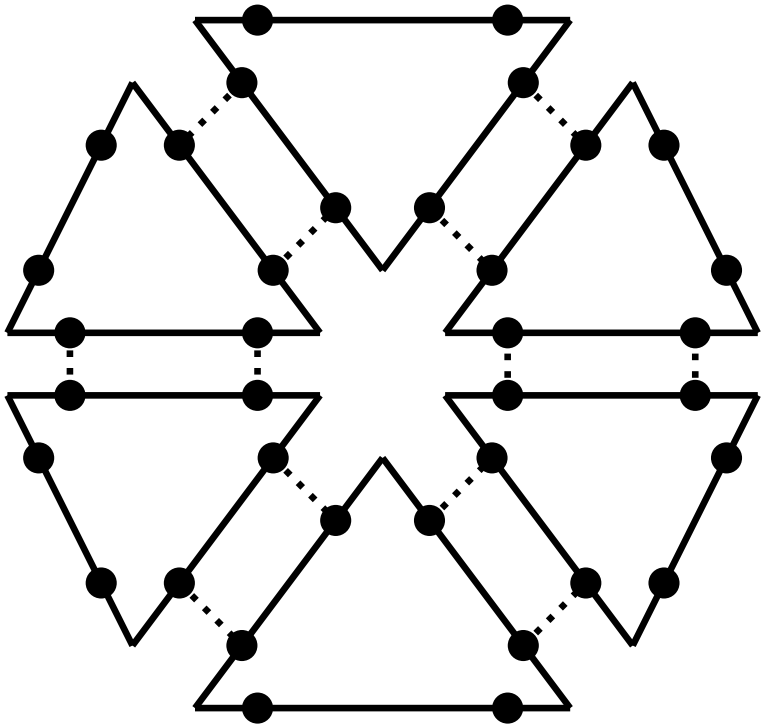

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- General tensor fields are analogous.

Vector Laplacian eigenvalue problems on surfaces

Hodge Laplacian (e.g. Maxwell)

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

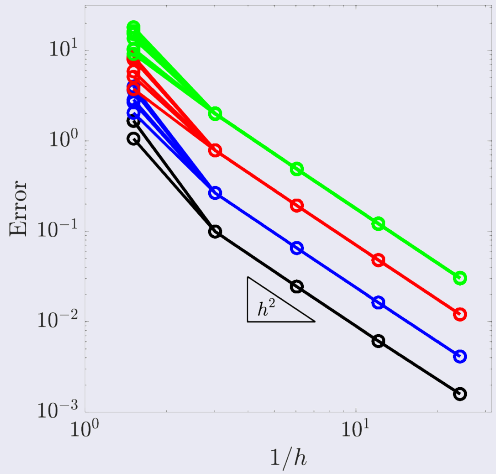
- Tangential continuity across edges suffices.
- Standard FEEC works.
- L^2 pairing suffices.

Bochner Laplacian (e.g. Stokes)

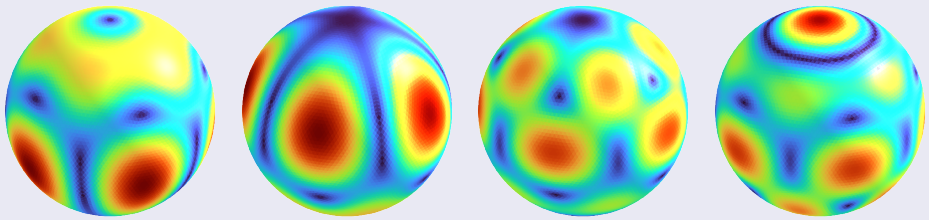
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

Bochner Laplacian on sphere using blow-up elements



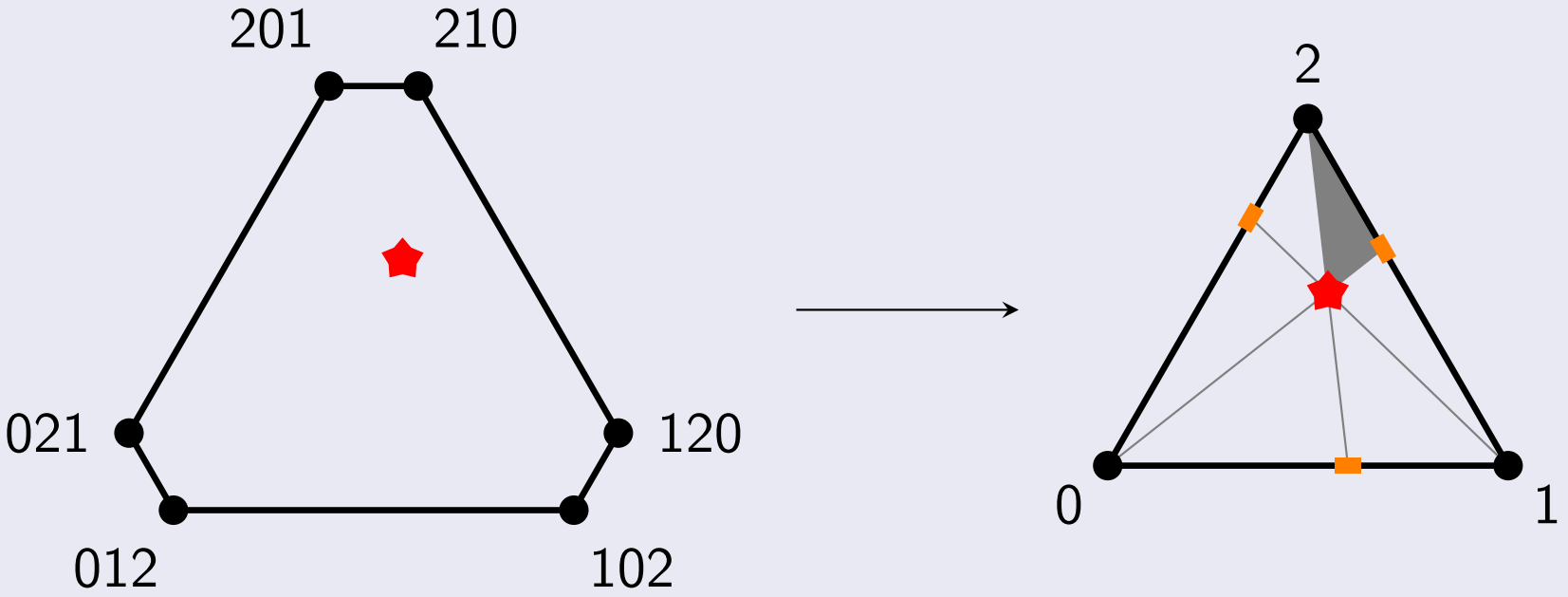
Eigenvalue error



Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

Blowing up the simplex

Configuration space of barycentric subdivisions



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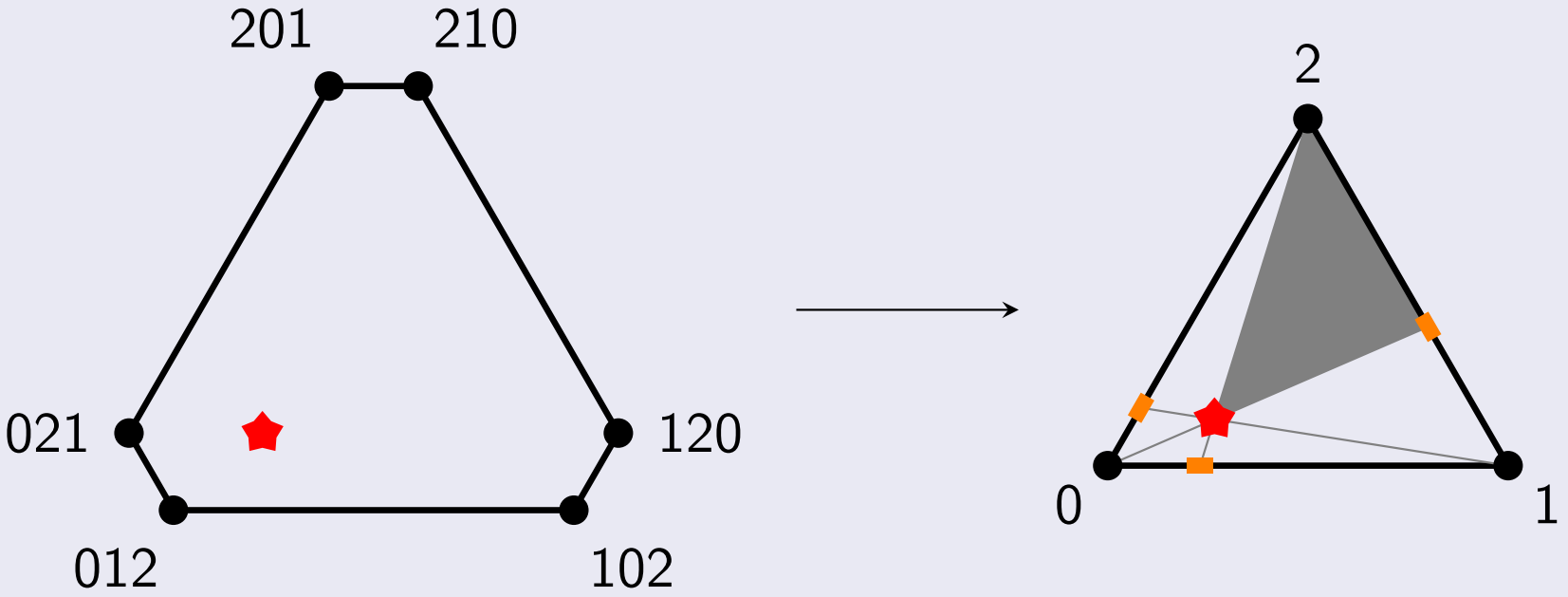

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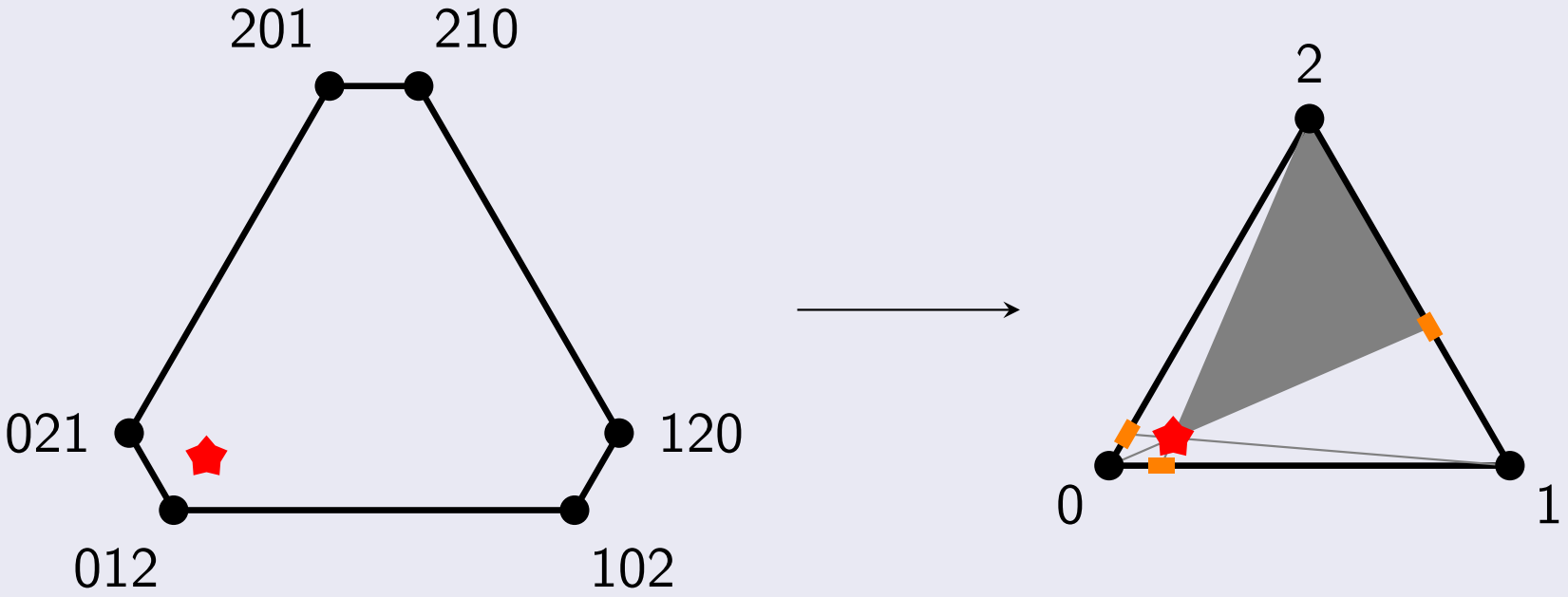

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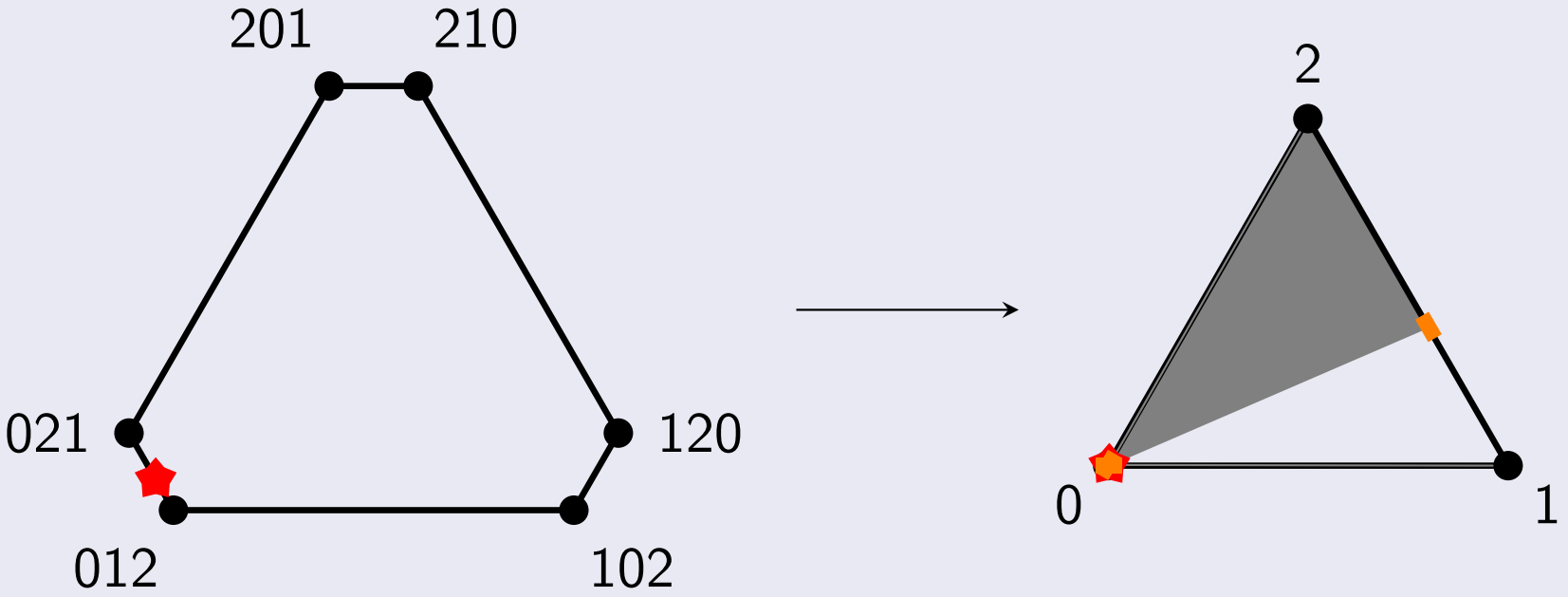

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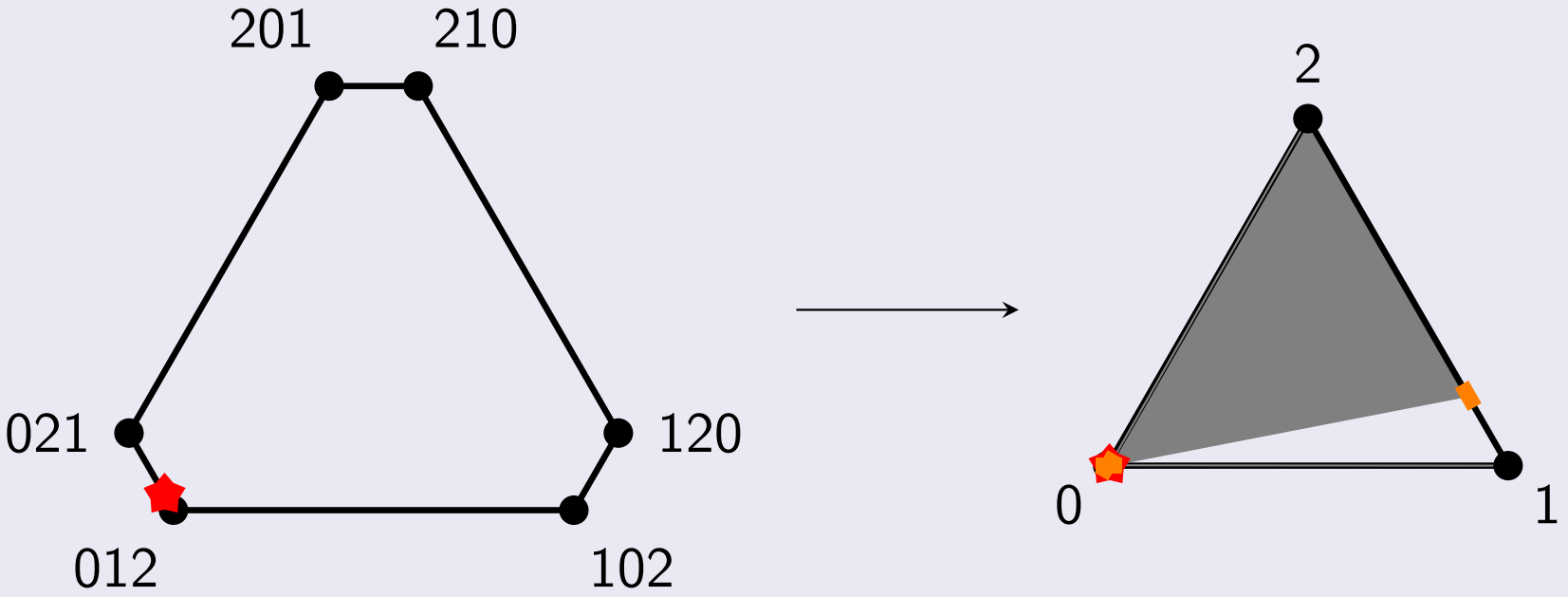

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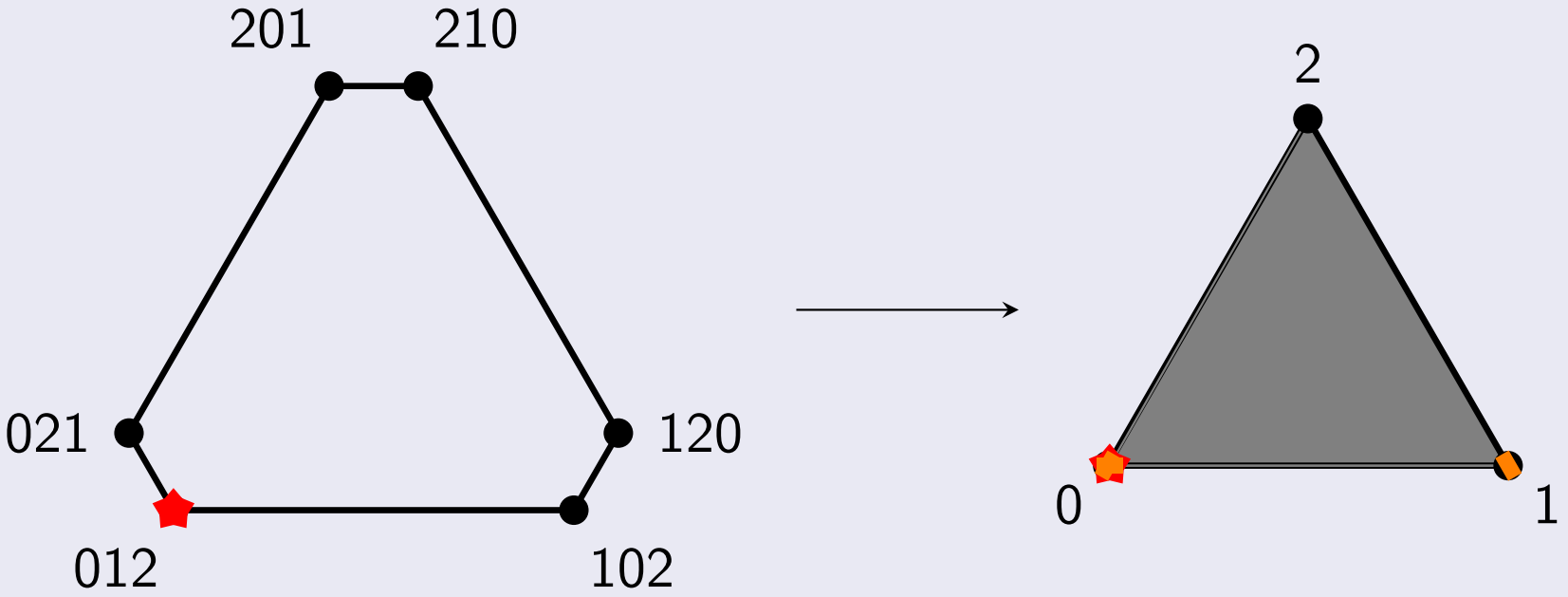

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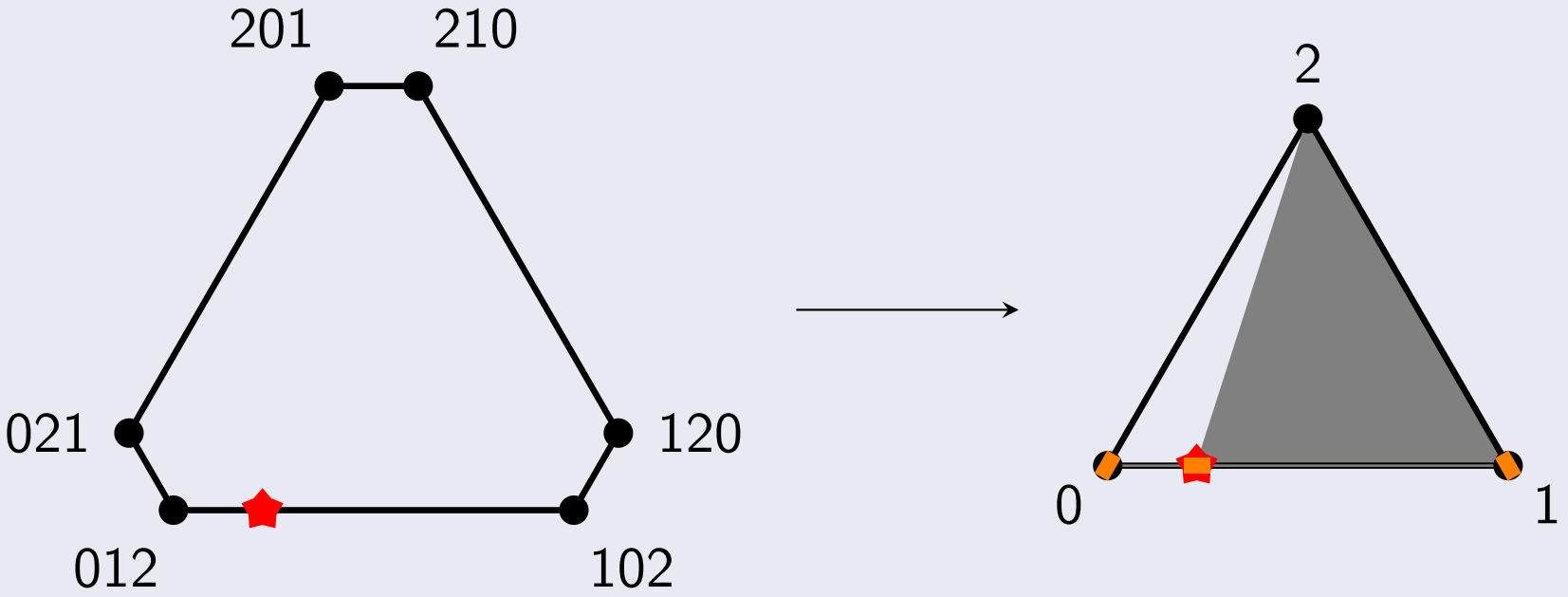

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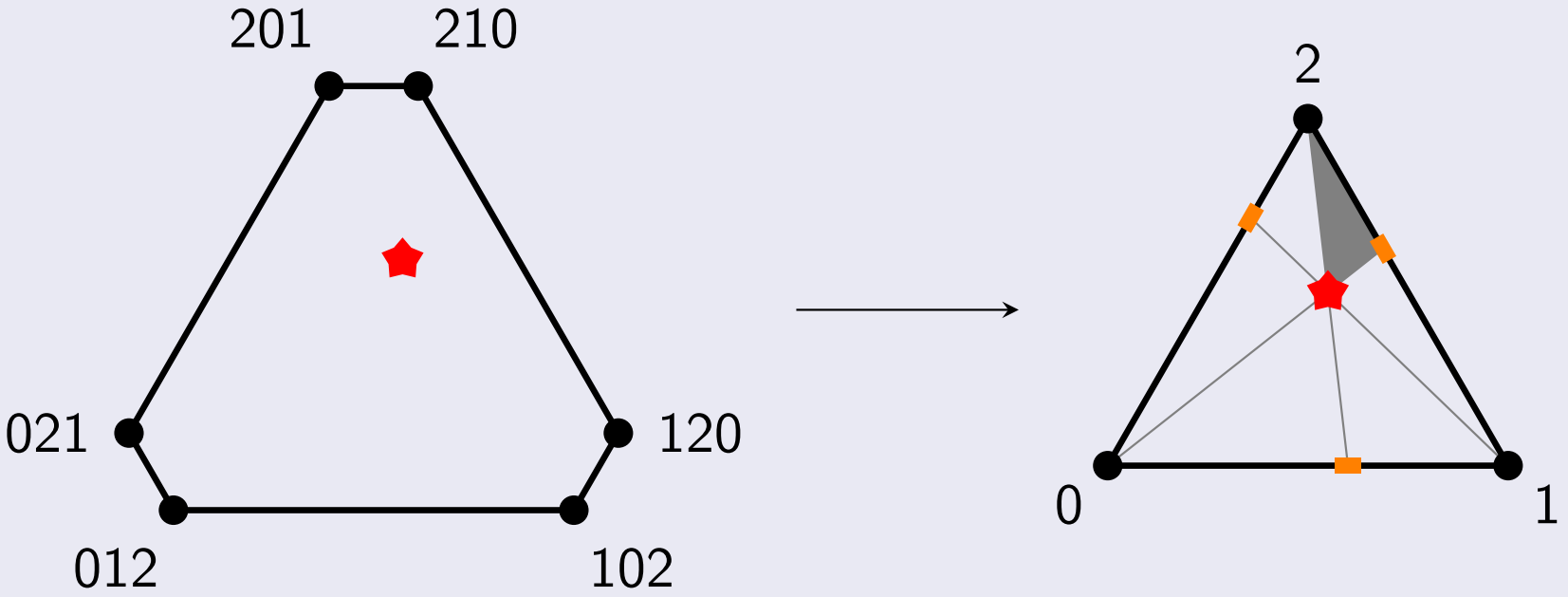

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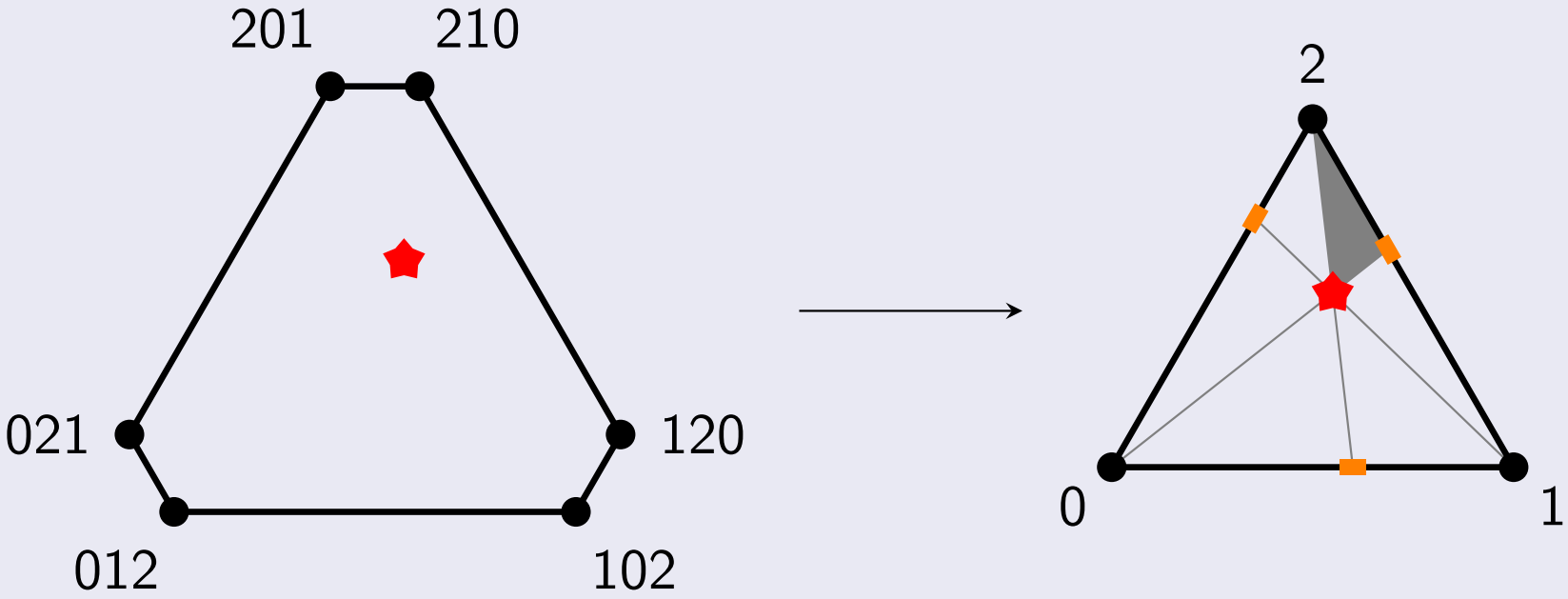

Smooth $\longleftrightarrow$ Smooth "in polar coordinates"

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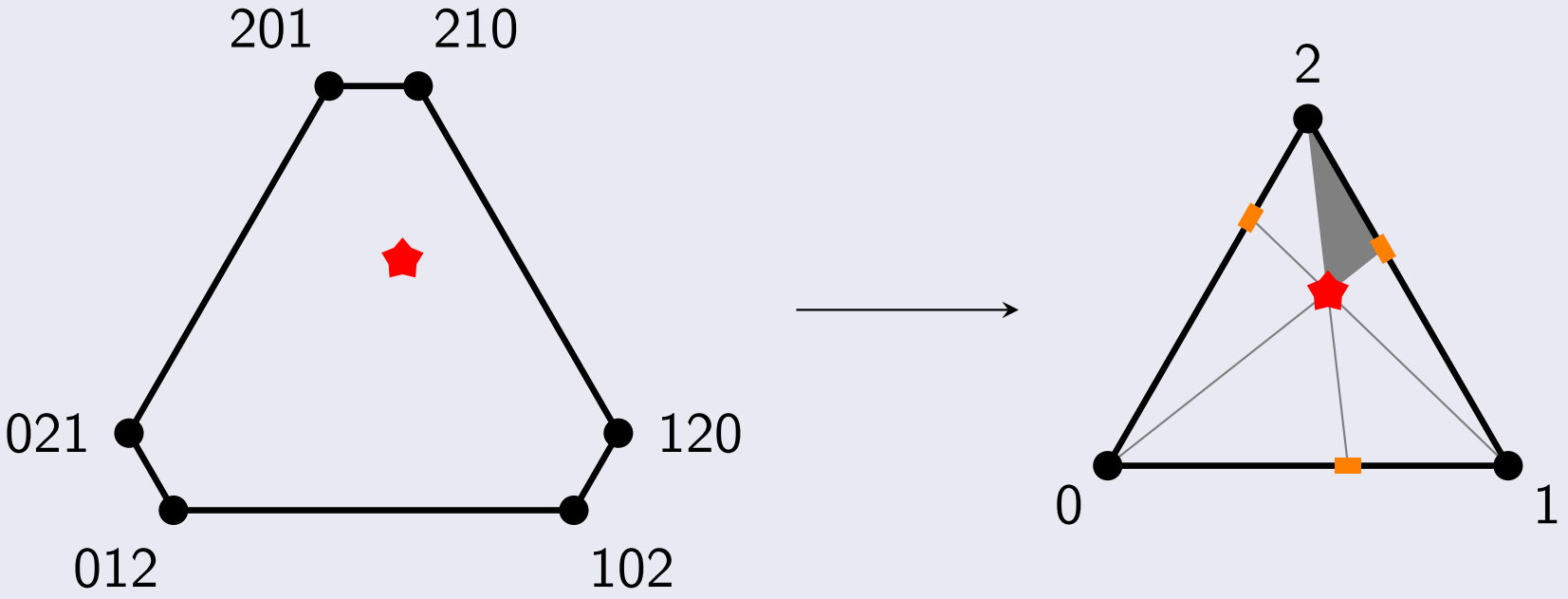

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- For the general theory of blowing up manifolds with corners, see Melrose, 1996.

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- Degrees of freedom of blow-up Whitney k-forms are integration over k-faces of the blow-up simplex.

What's next

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- Rational functions for Stokes problem (Guzmán, Neilan, 2014).

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- Original approach (—, Gawlik) via blow-ups and

$$\int_M \langle R_{\mathrm{dist}} v, w\rangle \wedge \beta := \int_M \langle \nabla v \wedge \nabla w\rangle \wedge \beta + \int_M \langle \nabla v, w\rangle \wedge d\beta$$

Section 2

Double forms

Differential forms corresponding to vector field $\langle M, N, P \rangle$

One-forms Λ^1

- $M\,dx + N\,dy + P\,dz$
- Restricted to the xy-plane $z = 0$:
  - $M\,dx + N\,dy$.
  - Tangential components.

Two-forms Λ^2

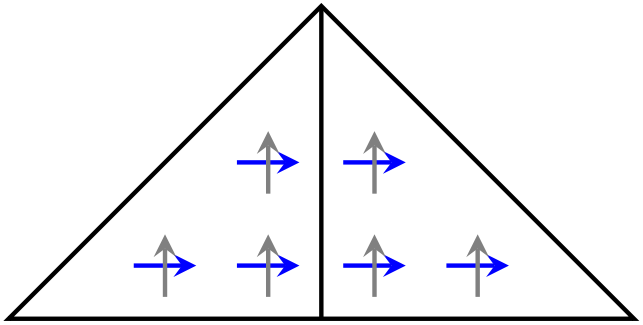
- $M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$.
- Restricted to the xy-plane $z = 0$:
  - $P\,dx \wedge dy$.
  - Normal component.

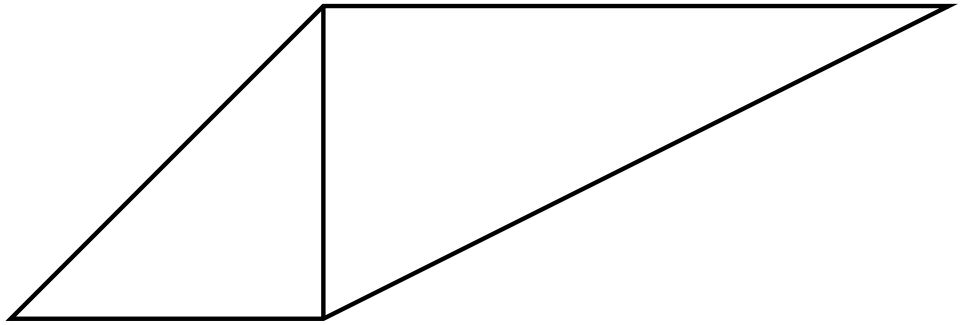
Continuity conditions

- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are $(n-1)$-forms.

How do "vector" fields transform?

- Transform as velocity (vector field): tangential and normal jumps.





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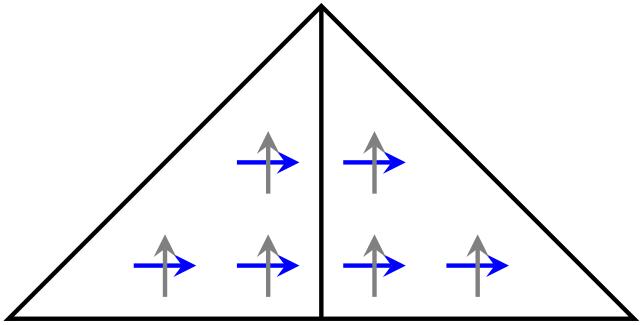

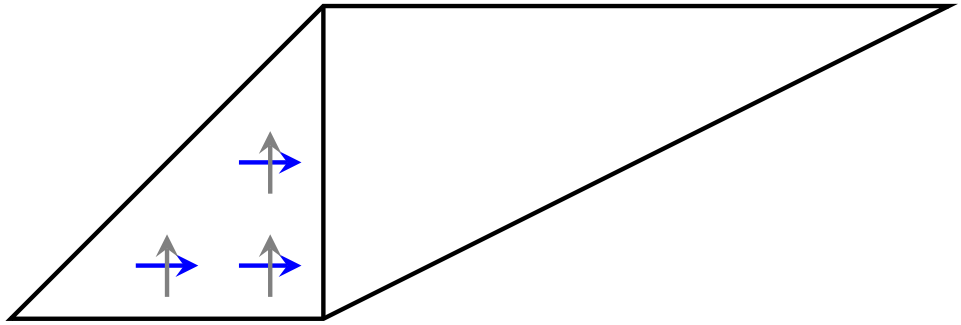

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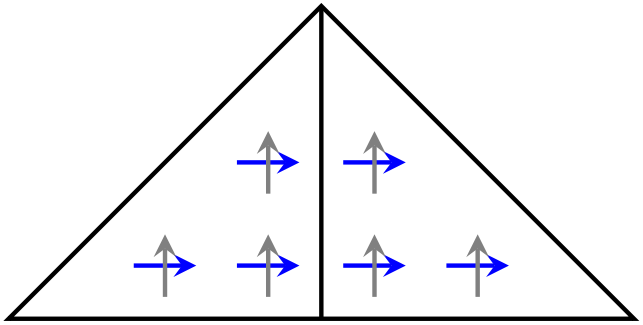

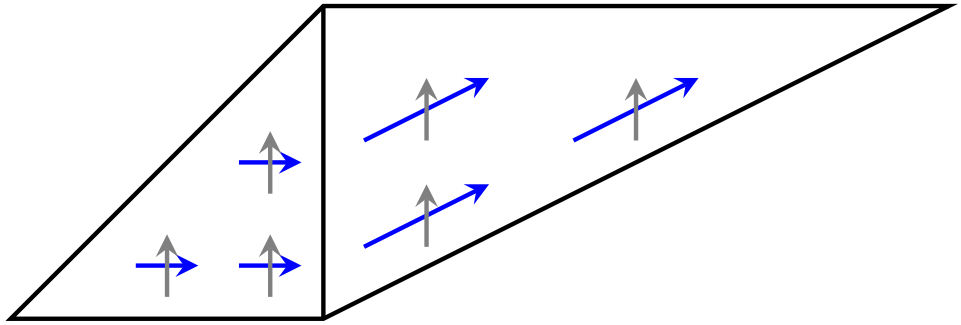

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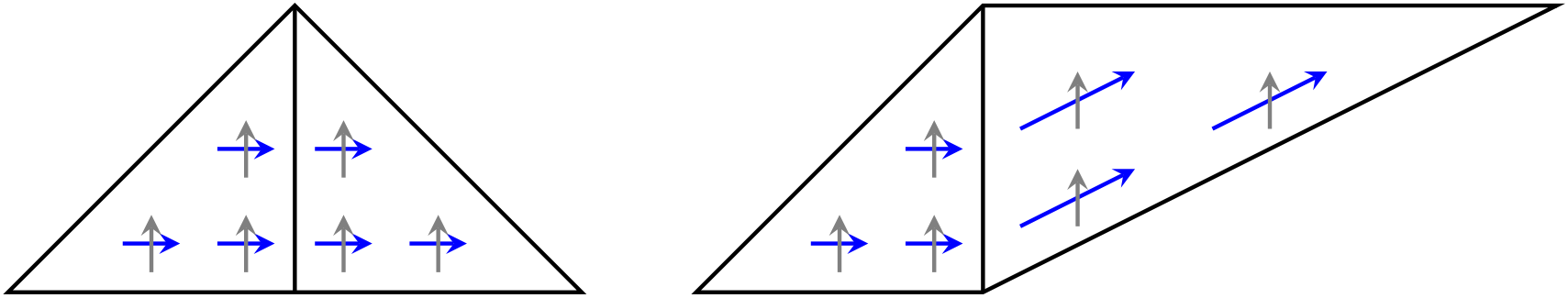

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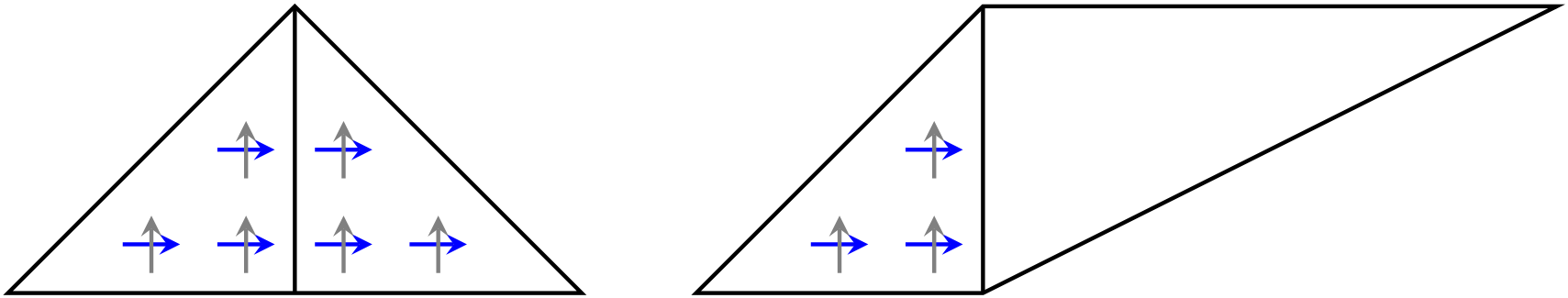

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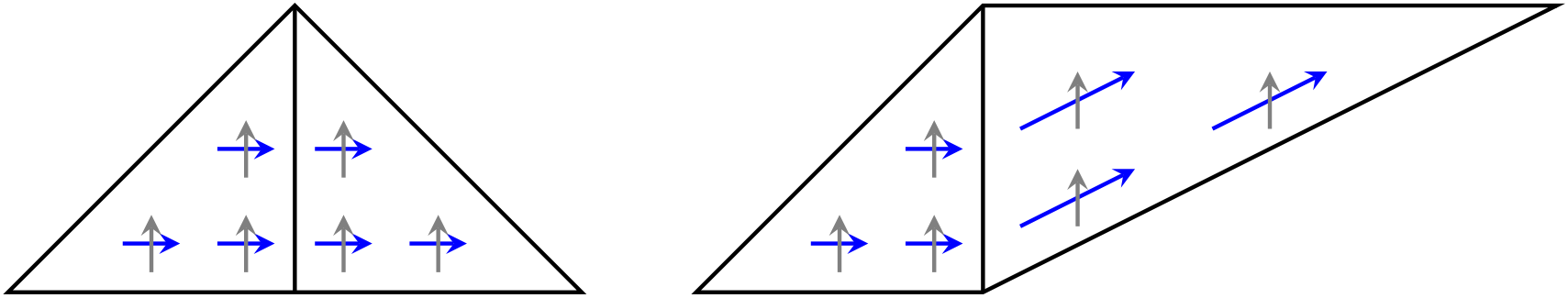

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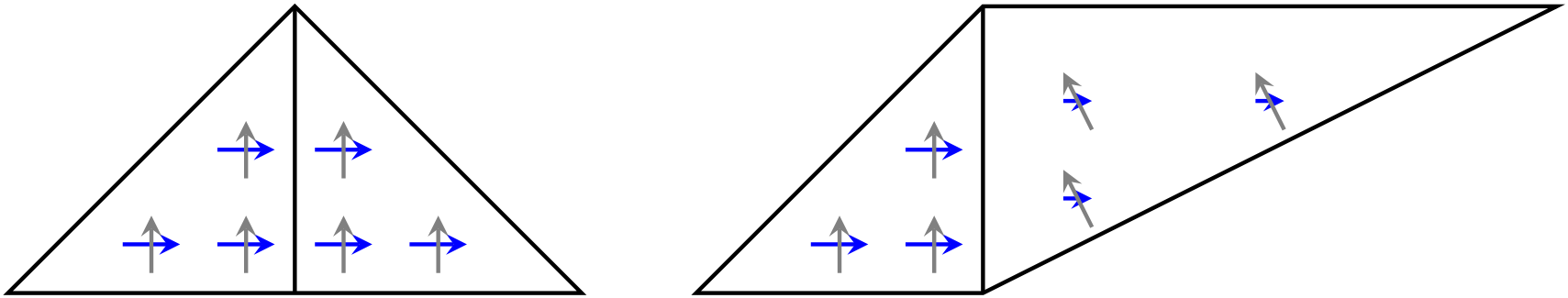

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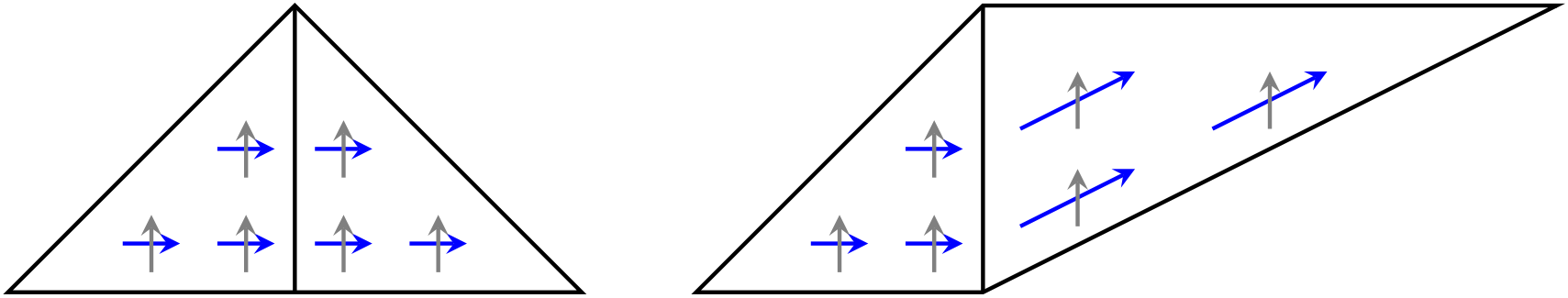

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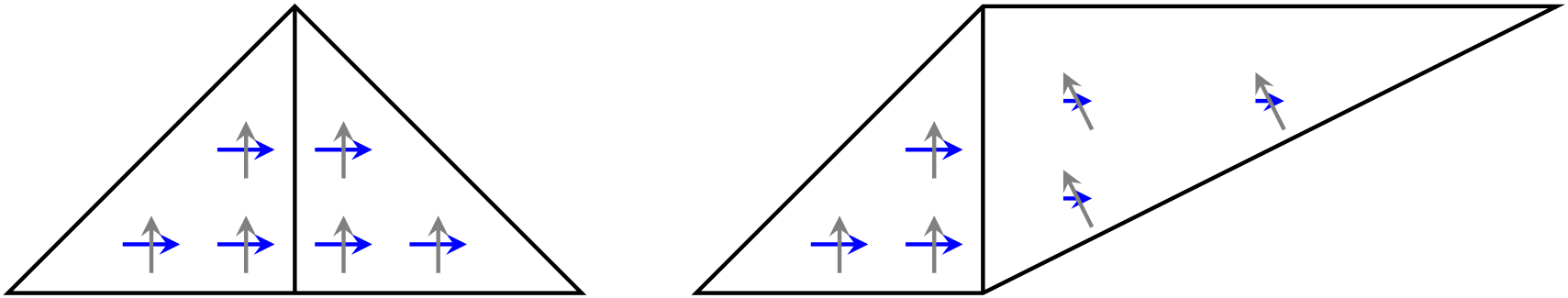

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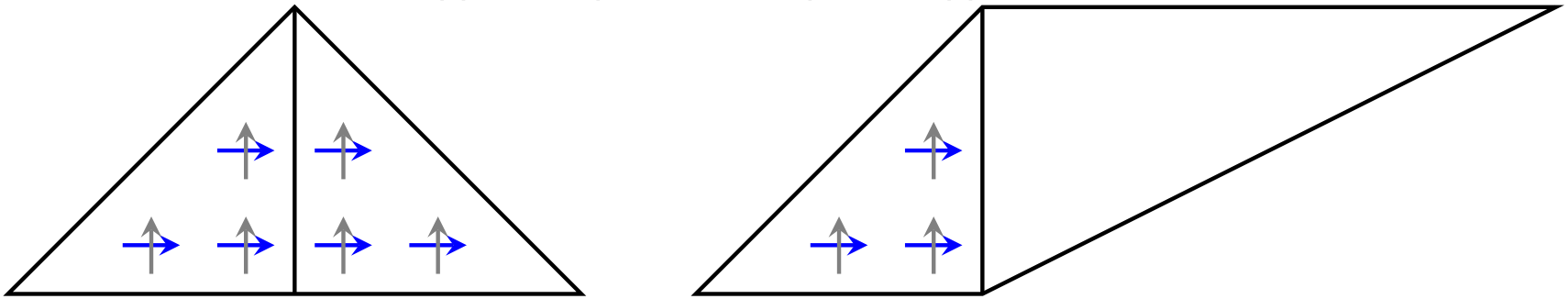

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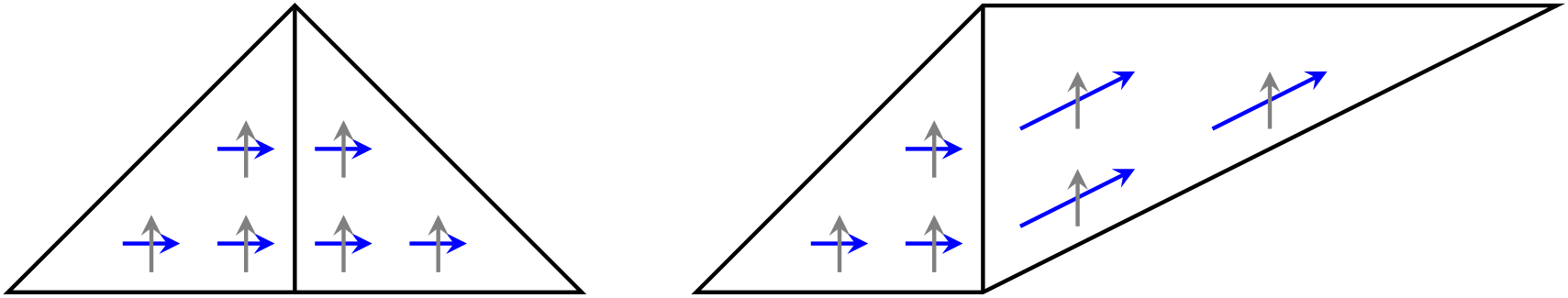

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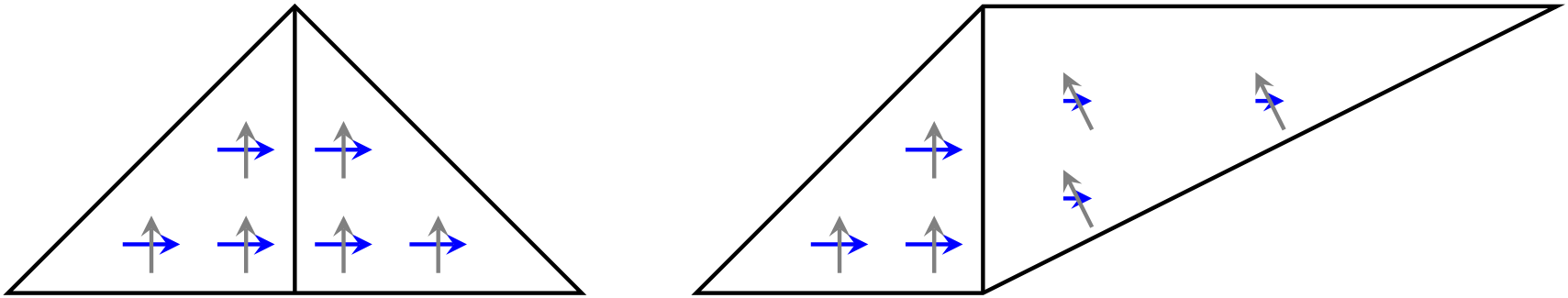

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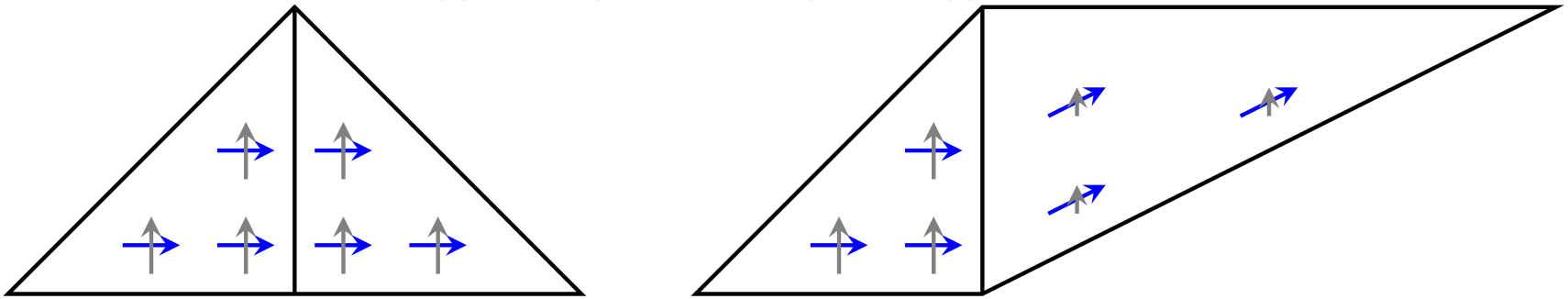

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- FEEC spaces are invariant under piecewise affine transformations of the mesh.

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- What about Finite Element Tensor Calculus (FETC)? Can we make affine-invariant spaces?

Extending FEEC to matrices and tensors

Vector fields ($\mathbb{R}^3$)

- Vector fields with tangential continuity are one-forms Λ^1.
- Vector fields with normal continuity are two-forms Λ^2.

Matrix fields ($\mathbb{R}^3 \otimes \mathbb{R}^3$)

- Matrix fields with tangential–tangential continuity are $(1,1)$-forms $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$.
- Matrix fields with normal–tangential continuity are $(2,1)$-forms $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$.
- Matrix fields with normal–normal continuity are $(2,2)$-forms $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$.

Applications

Strain/stress tensors

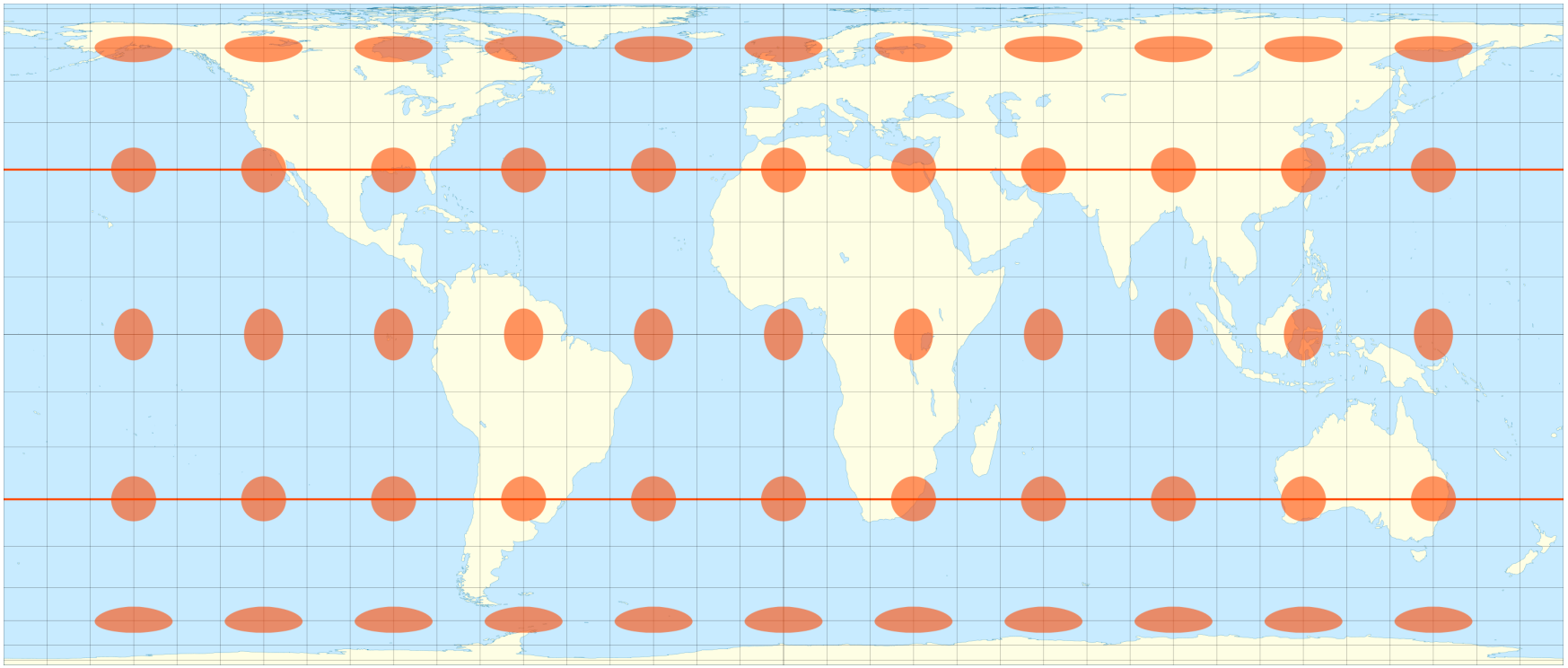
- Elasticity (objects deforming under stress)
- Fluid mechanics (Stokes equations)

Numerical geometry/relativity

- Riemannian/Minkowski metric
- Curvature tensor

Regge metrics $\Lambda_0^{1,1}$

Symmetric matrix fields with tangential–tangential continuity



Map credit: Wikipedia, Gaba

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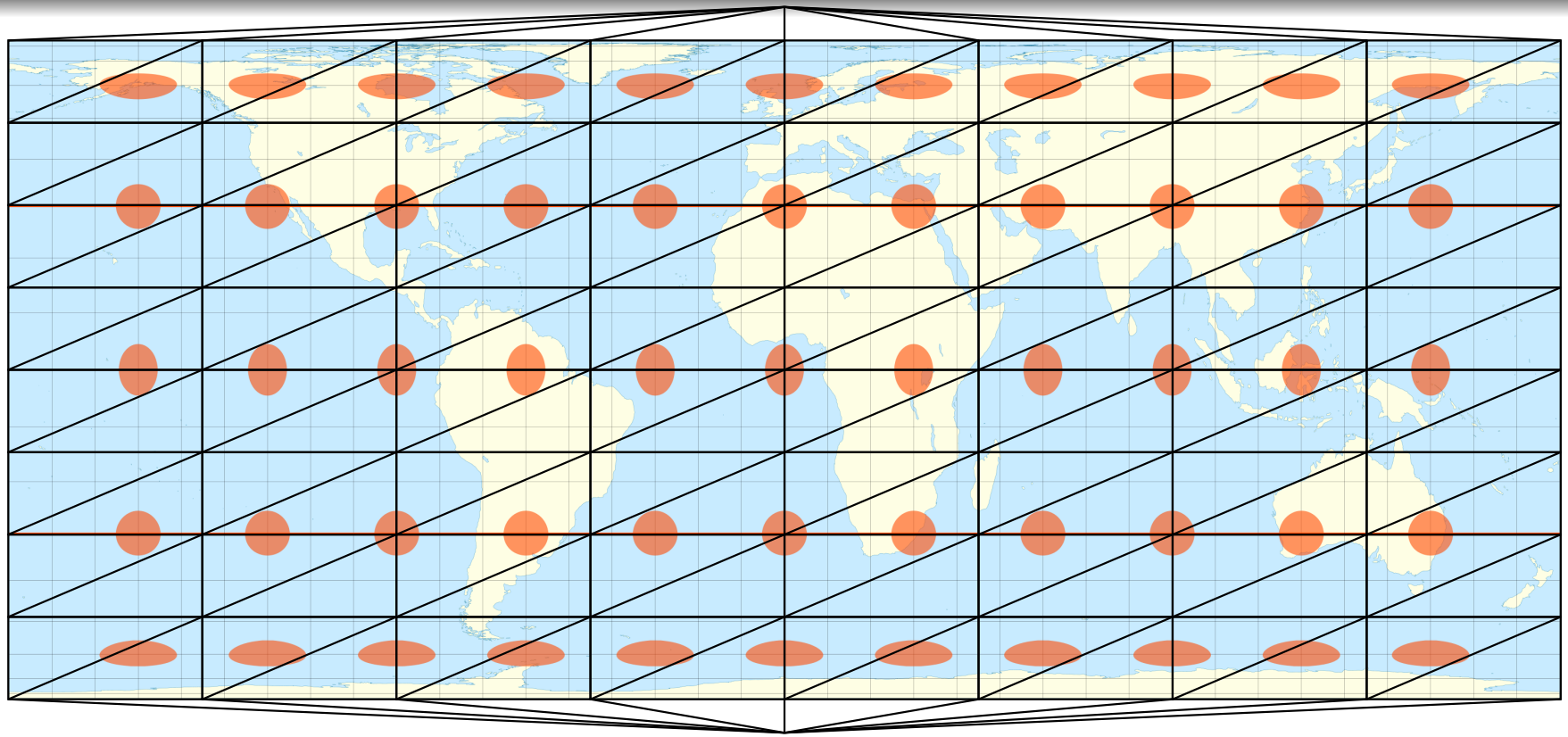

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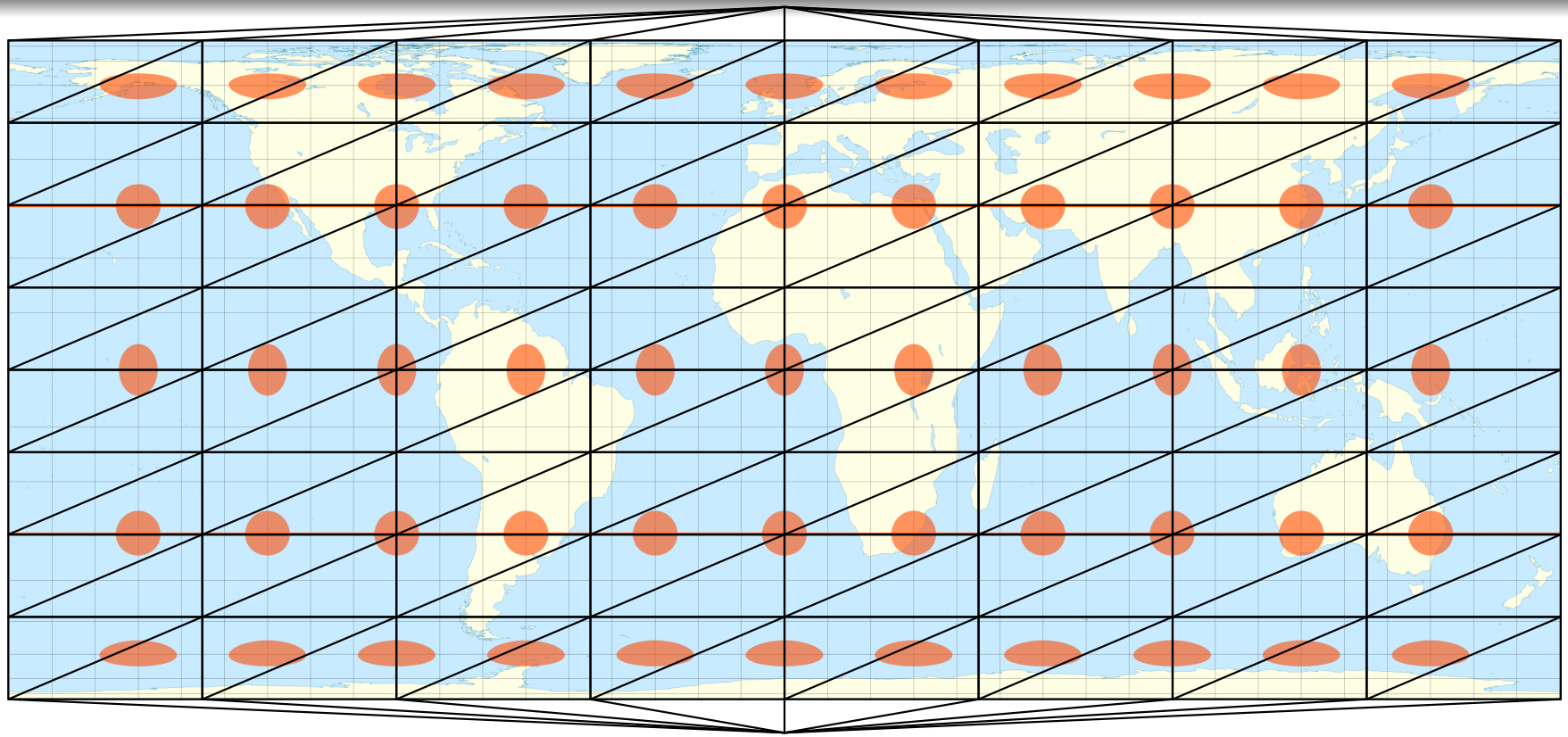


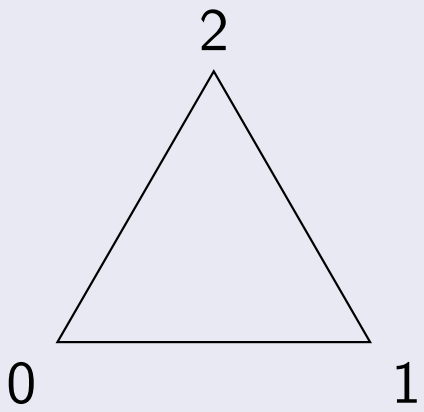
Regge finite elements

- Record the length of each edge.
- For each triangle, use the corresponding Euclidean metric.
- Get piecewise constant metric with tang.–tang. continuity.

Map credit: Wikipedia, Gaba

Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



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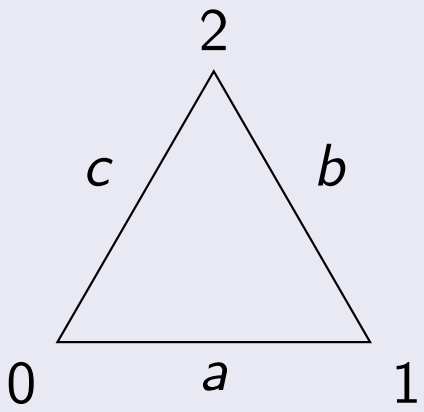

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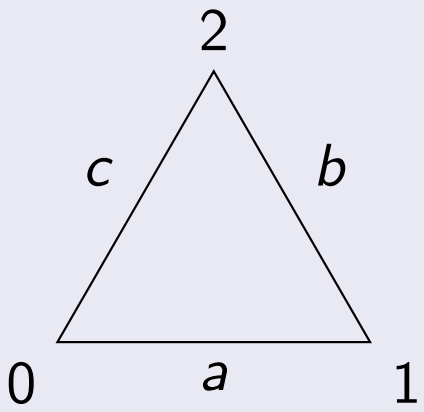

Regge metric:

$$
\begin{aligned}
g = &-\tfrac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0) \\
&-\tfrac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) \\
&-\tfrac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2)
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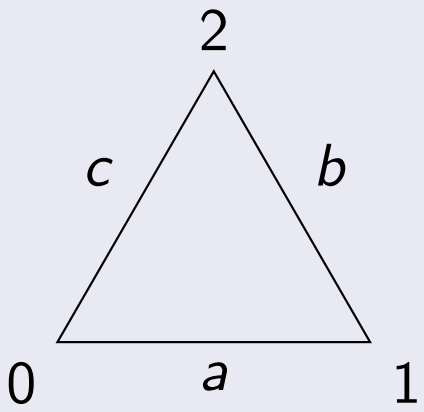

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Observations

- If $\mathbf{v}$ is the vector from vertex 0 to vertex 1, then

$$d\lambda_0(\mathbf{v}) = -1, \qquad d\lambda_1(\mathbf{v}) = 1, \qquad d\lambda_2(\mathbf{v}) = 0.$$

As desired:

$$g(\mathbf{v},\mathbf{v}) = -\tfrac{1}{2}a^2(-1-1) - \tfrac{1}{2}b^2(0+0) - \tfrac{1}{2}c^2(0+0) = a^2.$$

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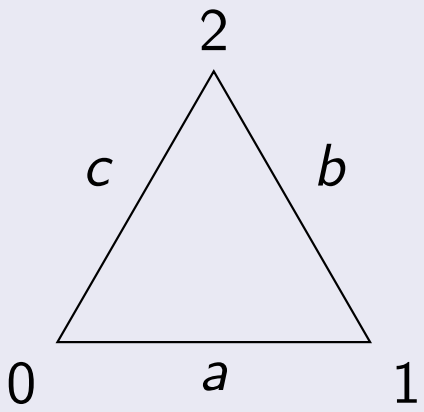

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- Crucial: $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ is zero on other edges.

Constant coefficient finite elements for bilinear forms

Geometrically decomposed bases for finite element spaces

- Each basis element φ must be associated to a face F of the triangulation, such that, for any other face G,

$$\varphi \text{ has nonzero trace on } G \Leftrightarrow G \geq F.$$

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Constant coefficient symmetric bilinear forms $\Lambda^{1,1}_{\text{sym}}$

- Regge's construction works in any dimension. To each edge ij, associate

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Constant coefficient antisymmetric bilinear forms $\Lambda^{1,1}_{\text{asym}}$

- Finite element spaces do not exist in dimension ≥ 3.
- In 3D, antisymmetric bilinear forms $\leftrightarrow$ vector fields with normal continuity.
- A nonzero constant vector field can't be tangent to three faces of a tetrahedron.

Affine-invariant subspaces of double forms

Theorem (Eigendecomposition of s^*s)

$$\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m, \qquad \max\{0, q-p\} \le m \le \min\{q, n-p\}.$$

Example

- $\Lambda^{1,1}_0$: Symmetric bilinear forms, $\varphi(X;Y) = \varphi(Y;X)$.
- $\Lambda^{1,1}_1$: Λ^2, antisymmetric bilinear forms, $\varphi(X;Y) = -\varphi(Y;X)$.

- $\Lambda^{2,1}_0$: spanned by $\alpha \otimes \beta$ such that $\alpha \wedge \beta = 0$.
  - Matrix proxy in 3D: trace-free matrices.
- $\Lambda^{2,1}_1$: Λ^3.
  - Matrix proxy in 3D: multiples of the identity matrix.

- $\Lambda^{2,2}_0$: Symmetric, satisfying the algebraic Bianchi identity.
  - Riemann curvature tensor.
- $\Lambda^{2,2}_1$: Antisymmetric, $\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$.
- $\Lambda^{2,2}_2$: Λ^4.

The decomposition respects the Bianchi sum operator s

$s\colon \Lambda_m^{p,q} \xrightarrow{\sim} \Lambda_{m-1}^{p-1,q-1}$

$$0 \xrightarrow{s} \Lambda^{0,4} \xrightarrow{s} \Lambda^{1,3} \xrightarrow{s} \Lambda^{2,2} \xrightarrow{s} \Lambda^{3,1} \xrightarrow{s} \Lambda^{4,0} \xrightarrow{s} 0$$

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$$
\begin{array}{ccccccccccccc}
 & & & & & & \Lambda_0^{2,2} & & & & & & \\
 & & & & & & \oplus & & & & & & \\
 & & & & \Lambda_2^{1,3} & & \Lambda_1^{2,2} & & \Lambda_0^{3,1} & & & & \\
 & & & & \oplus & & \oplus & & \oplus & & & & \\
 & & \Lambda_4^{0,4} & & \Lambda_3^{1,3} & & \Lambda_2^{2,2} & & \Lambda_1^{3,1} & & \Lambda_0^{4,0} & & \\
 & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \\
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 & & & & & & \oplus & & & & & & \\
 & & 0 & \xrightarrow{s} & \Lambda_2^{1,3} & \xrightarrow[\sim]{s} & \Lambda_1^{2,2} & \xrightarrow[\sim]{s} & \Lambda_0^{3,1} & \xrightarrow{s} & 0 & & \\
 & & & & \oplus & & \oplus & & \oplus & & & & \\
0 & \xrightarrow{s} & \Lambda_4^{0,4} & \xrightarrow[\sim]{s} & \Lambda_3^{1,3} & \xrightarrow[\sim]{s} & \Lambda_2^{2,2} & \xrightarrow[\sim]{s} & \Lambda_1^{3,1} & \xrightarrow[\sim]{s} & \Lambda_0^{4,0} & \xrightarrow{s} & 0 \\
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\end{array}
$$

Short proof of injectivity/surjectivity of the Bianchi sum s

Proposition

The operator $s\colon \Lambda^{p,q} \to \Lambda^{p+1,q-1}$ *is injective if* $p < q$ *and surjective if* $p+1 > q-1$.

Short proof of injectivity/surjectivity of the Bianchi sum s

Proposition

The operator $s\colon \Lambda^{p,q} \to \Lambda^{p+1,q-1}$ *is injective if* $p < q$ *and surjective if* $p+1 > q-1$.

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Finite element spaces

Theorem (—, Gawlik)

Apart from $\Lambda_q^{p,q} \cong \Lambda^{p+q}$ with constant coefficients, there is a finite element space for every natural space of double forms $\Lambda_m^{p,q}$ with polynomial coefficients of any degree (including zero).

Example (Constant coefficient spaces)

- $\Lambda_0^{1,1}$: symmetric matrices with tangential–tangential continuity (Regge, 1961; Christiansen, 2004).
  - Higher order: (Li, 2018).
- $\Lambda_0^{2,1}$ in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda_0^{2,2}$ in 3D: symmetric matrices with normal–normal continuity (Pechstein and Schöberl, 2011).
- $\Lambda_0^{2,2}$ (or $\Lambda_0^{n-2,n-2}$) in any dimension: finite elements for the Riemann curvature tensor.

Degrees of freedom for constant coefficient spaces

	d						
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	**1**	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	**2**	0	0	0	0
$\Lambda_0^{2,2}$	0	0	**1**	**2**	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	**3**	0	0	0
$\Lambda_0^{3,2}$	0	0	0	**3**	**5**	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	**4**	0	0
$\Lambda_0^{3,3}$	0	0	0	**1**	**5**	**5**	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	**6**	**9**	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	**5**	0

Table: Number of degrees of freedom for $\Lambda_m^{p,q}$ associated to a face of the triangulation of dimension d is $\frac{p-q+2m+1}{p+m+1}\binom{d+1}{q-m}\binom{q-m-1}{d-p-m}$.

More on algebraic curvature tensors $\Lambda_0^{2,2}$

Joint with Lily DiPaulo

The space $\Lambda_0^{2,2}$

- Symmetric $(2,2)$-forms satisfying the Bianchi identity.
- $\Lambda_0^{2,2}$ is spanned by $\alpha \odot \beta$ where $\alpha, \beta \in \Lambda^2$ and $\alpha \wedge \beta = 0$.

Finite element spaces

- Construct bases for constant coefficient spaces using (—, Gawlik)
- Generalize to higher order similarly to Li's work on Regge finite elements.

Regge finite elements $\mathcal{P}_r\Lambda_0^{1,1}$ (symmetric bilinear forms)

Constant coefficient space $\Lambda_0^{1,1}$

- For i and j distinct vertices, associate $d\lambda_i \odot d\lambda_j$ to edge ij.
- These forms are a basis for the space $\Lambda_0^{1,1}$ of symmetric bilinear forms with constant coefficients.

Higher order spaces $\mathcal{P}_r\Lambda_0^{1,1}$ (Li)

- For a multiindex I, let λ^I be the corresponding monomial, and let $\operatorname{supp} I$ denote the set of vertices whose corresponding exponent is at least one in λ^I.
  - e.g. if $\lambda^I = \lambda_0^5\lambda_3^4$ then $\operatorname{supp} I = \{0, 3\}$.
- Associate $\lambda^I\, d\lambda_i \odot d\lambda_j$ to the face with vertices $\{i, j\} \cup \operatorname{supp} I$.
- These forms are a basis for $\mathcal{P}_r\Lambda_0^{1,1}$ because the monomials are a basis for $\mathcal{P}_r$ and the $d\lambda_i \odot d\lambda_j$ are a basis for $\Lambda_0^{1,1}$.

Constant coefficient space $\Lambda_0^{2,2}$

- Let $d\lambda_{ij} := d\lambda_i \wedge d\lambda_j$.
- To each two-dimensional face ijk, associate

$$\beta_{ijk} := d\lambda_{ij} \odot d\lambda_{jk} + d\lambda_{jk} \odot d\lambda_{ki} + d\lambda_{ki} \odot d\lambda_{ij}$$

- To each three-dimensional face $ijkl$, associate

$$\gamma_{iklj} := d\lambda_{il} \odot d\lambda_{jk} - d\lambda_{ij} \odot d\lambda_{kl},$$
$$\gamma_{iljk} := d\lambda_{ij} \odot d\lambda_{kl} - d\lambda_{ik} \odot d\lambda_{lj}.$$

- These forms are a basis for the space $\Lambda_0^{2,2}$ of algebraic curvature tensors with constant coefficients.
- These formulas can be derived from the representation theory of the symmetric group (Young diagrams), following (—, Gawlik).

Higher order algebraic curvature tensors $\mathcal{P}_r\Lambda_0^{2,2}$

Constant coefficient space $\Lambda_0^{2,2}$

$$\begin{aligned}\beta_{ijk} &:= d\lambda_{ij} \odot d\lambda_{jk} + d\lambda_{jk} \odot d\lambda_{ki} + d\lambda_{ki} \odot d\lambda_{ij},\\ \gamma_{iklj} &:= d\lambda_{il} \odot d\lambda_{jk} - d\lambda_{ij} \odot d\lambda_{kl},\\ \gamma_{iljk} &:= d\lambda_{ij} \odot d\lambda_{kl} - d\lambda_{ik} \odot d\lambda_{lj}.\end{aligned}$$

Higher order space $\mathcal{P}_r\Lambda_0^{2,2}$

- Associate $\lambda^I\beta_{ijk}$ to the face with vertices $\{i,j,k\} \cup \operatorname{supp} I$.
- Associate $\lambda^I\gamma_{iklj}$ and $\lambda^I\gamma_{iljk}$ to the face with vertices $\{i,j,k,l\} \cup \operatorname{supp} I$.
- These forms are a geometrically decomposed basis for $\mathcal{P}_r\Lambda_0^{2,2}$.

What's next?

Explicit bases for arbitrary double forms

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- Use representation theory to write down explicit bases for $\Lambda_0^{p,q}$ for general p and q.

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Arbitrary covariant tensors with arbitrary symmetries

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Distributional Riemann curvature, joint with Gawlik and Neunteufel

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Distributional Riemann curvature, joint with Gawlik and Neunteufel

- The natural test function space for the distributional Riemann curvature is $\Lambda_0^{n-2,n-2}$.

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik

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