Blow-up finite elements

Yakov Berchenko-Kogan, joint with Evan Gawlik

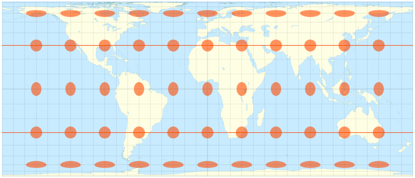
Florida Institute of Technology
Supported by NSF DMS-2411209

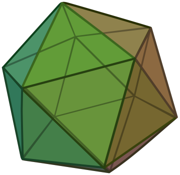
July 1, 2026

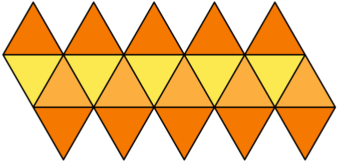
Surfaces: extrinsic vs. intrinsic

Extrinsic

Intrinsic







Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

Four images from Wikipedia

Discretizing continuous tangent vector fields on surfaces is "impossible"

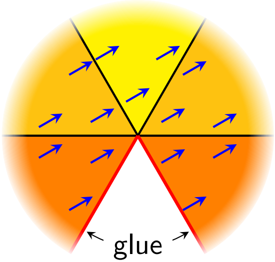
Vector field requirements

1. Tangent to the surface.
2. No jump across edges.
3. Continuous on each triangle.

Let's try!

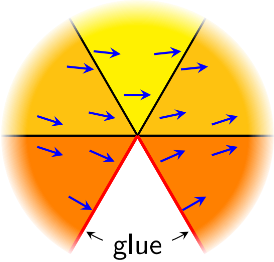
- Zoom in on a vertex of the icosahedron.
- What do we see?

continuous elements



discontinuous across red edge

blow-up elements



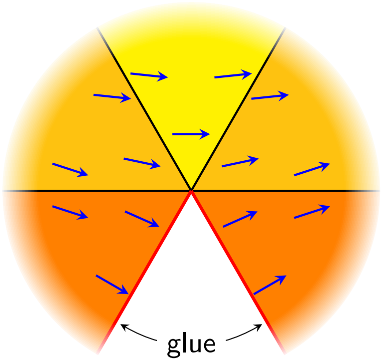
continuous across all edges

Approaches to discretizing surface vector fields

- Break tangentiality to surface: Use vector fields in ambient space, penalize component normal to surface.
  - Unnecessary data (normal component).
  - Extrinsic.
- Break continuity across edges:
  - For problems like electromagnetism ($H(\text{curl})$ (or $H(\text{div})$)), use FEEC (BDM, RT) elements.
    - Jump in component normal to edge but not in component tangent to edge (or vice versa).
    - Intrinsic via FEEC (differential forms) formulation.
    - In this context, edge jumps are actually good (full continuity is bad), see (Arnold, Falk, Winther, 2010).
  - In contrast, for problems like fluid flow (H^1), edge jumps are bad.
    - One approach: Use FEEC anyways and penalize jump across edges.
    - Demlow, Neilan approach: At each vertex, project to a plane (no angle defect), and enforce continuity in that plane.
- Break continuity at vertices: Blow-up elements (—, Gawlik).
  - Intrinsic.
  - Discontinuities are *only* where the problem is (vertices with angle defect).

Break continuity at vertices

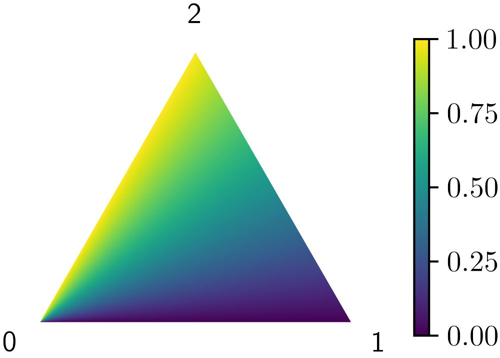
blow-up elements



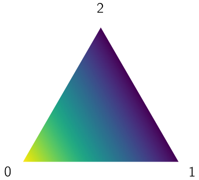
continuous across all edges

discontinuous at vertices

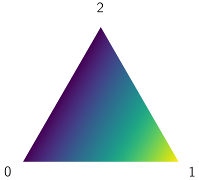
A simplicial analogue of the angular coordinate



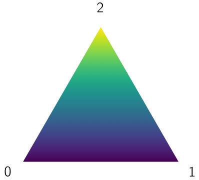
$$\frac{\lambda_2}{\lambda_1 + \lambda_2}$$



λ_0

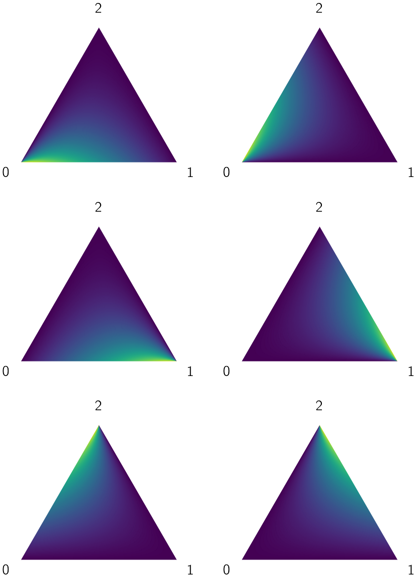


λ_1



λ_2

Blow-up $b\mathcal{P}_1$ shape functions



$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

New?

Shape function

$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}\cdot\frac{\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{\lambda_2}{\lambda_2}.$$

Earlier appearances

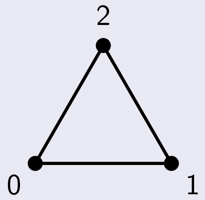
- Geometric invariants (Chen, 1957).
- Horse betting (Harville, 1973).
- Intersection homology (Brasselet, Goresky, MacPherson, 1991; Bendiffalah, 1995).

Poisson processes (—, Gawlik)

- Three radiation sources with rates λ_0, λ_1, and λ_2.
- Let t_0, t_1, t_2 be the times when the respective radiation sources produce their first particle.
- ψ_{012} is the probability that $t_0 \le t_1 \le t_2$.

Degrees of freedom

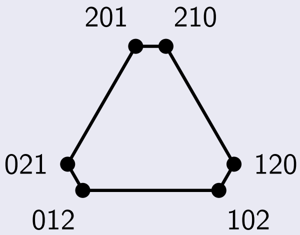
Classical Lagrange $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1.$

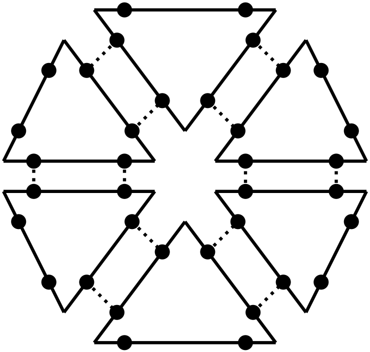
- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
- $2 : \lambda_2 = 1 \Leftrightarrow \lambda_0 = \lambda_1 = 0$

Blow-up $b\mathcal{P}_1$



- $012 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $120 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_0 \to 0}$
- $201 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $021 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $102 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $210 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_0 \to 0}$

Global spaces



Blow-up finite elements

- Scalar fields: we place a number at each dot.
- Vector fields: we place two numbers at each dot, for the tangential and normal components, respectively.
  - Enforce continuity for both components, yielding full continuity across edges.
- Matrix fields: At each dot, we record the tangential–tangential component, the tangential–normal component, etc.
  - Can impose conditions on the components such as symmetry, trace-free, etc.
  - Can enforce continuity for all components or just some of them.
- General tensor fields are analogous.

Vector Laplacian eigenvalue problems on surfaces

Hodge Laplacian (e.g. Maxwell)

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

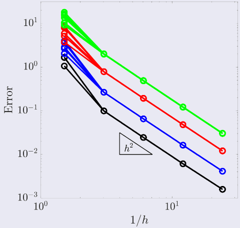
- Tangential continuity across edges suffices.
- Standard FEEC works.
- L^2 pairing suffices.

Bochner Laplacian (e.g. Stokes)

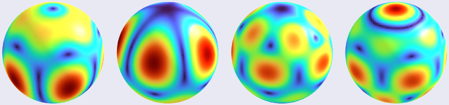
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

Bochner Laplacian on sphere using blow-up elements



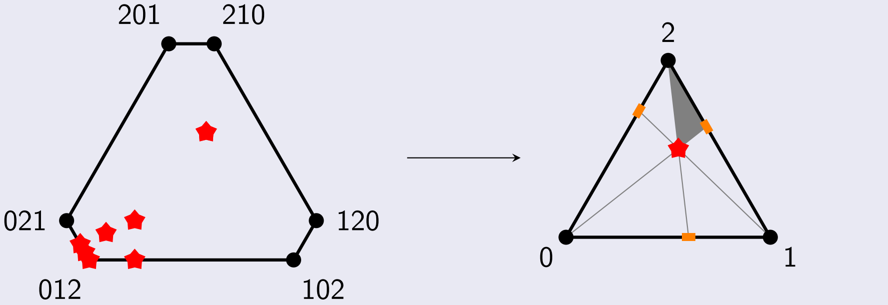
Eigenvalue error



Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

Blowing up the simplex

Configuration space of barycentric subdivisions



Smooth $\longleftrightarrow$ Smooth "in polar coordinates"

- Another interpretation of $\psi_{012} = \lambda_0 \frac{\lambda_1}{\lambda_1+\lambda_2}$: the relative area of the shaded region.
- For the general theory of blowing up manifolds with corners, see Melrose, 1996.

H^1 nonconforming

Blow-up fields are not in H^1

- Near vertices, blow-up scalar fields behave like θ.
- $\operatorname{grad}\theta$ grows like $\frac{1}{r}$.
- $\frac{1}{r} \in L^{2-\varepsilon}$ (or $H^{-\varepsilon}$) but not in L^2.

Impossibility of H^1 discretization of tangent vector fields

- Curvature can be computed via second derivatives of vector fields.
- Curvature is delta functions at vertices times angle defect.
- Delta functions are in $H^{-1-\varepsilon}$ but not in H^{-1}.

Bochner Laplacian

- Weak formulation $\int \langle \nabla u, \nabla v \rangle = \lambda \int \langle u, v \rangle$.
- $u, v \notin H^1$ means left-hand side is infinite, though it becomes finite if we remove a small neighborhood of the vertices.
- Use quadrature rule that doesn't evaluate at vertices, and it works?

There's more

So far in this talk

- Lowest order blow-up scalar fields in two dimensions, $b\mathcal{P}_1(T^2)$,
  - including tensor fields with components in $b\mathcal{P}_1(T^2)$.

Our paper

- Differential complex of blow-up Whitney k-forms in any dimension, $b\mathcal{P}_1^-\Lambda^k(T^n)$.
  - Shape functions previously studied in (Brasselet, Goresky, MacPherson, 1991), called shadow forms.
- ~~Higher-order blow-up scalar fields $b\mathcal{P}_r(T^n)$.~~
- A surprising connection to arrival times of Poisson processes, yielding simpler computations.
- Degrees of freedom of blow-up Whitney k-forms are integration over k-faces of the blow-up simplex.

What's next

Higher order blow-up scalar fields bP_r.

- Fluid flow (surface Stokes complex): blow-up generalization of P_2–P_1 velocity–presure pair. (See also Demlow, Neilan and Kone, Neilan, Poling for extrinsic method.)
- Implement in NGSolve, joint with undergraduate Aidan Nelappana.

Connections to other uses of rational functions?

- Rational functions for Stokes problem (Guzmán, Neilan, 2014).
- Duffy transform, Gregory patches, . . .

Distributional Riemann curvature

- Via blow-ups and frames: (Gawlik, McKee, 2026).
- Original approach (—, Gawlik) via blow-ups and

$$\int_M \langle R_{\mathrm{dist}} v, w\rangle \wedge \beta := \int_M \langle \nabla v \wedge \nabla w\rangle \wedge \beta + \int_M \langle \nabla v, w\rangle \wedge d\beta$$

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik
Blow-up Whitney forms, shadow forms, and Poisson processes.
Results in Applied Mathematics, special issue on Hilbert complexes, Paper No. 100529, 2025.

J. P. Brasselet, M. Goresky, and R. MacPherson.
Simplicial differential forms with poles.
Amer. J. Math., 113(6):1019–1052, 1991.

Supported by NSF DMS-2411209.