Blow-up finite elements

Yakov Berchenko-Kogan, joint with Evan Gawlik

Florida Institute of Technology
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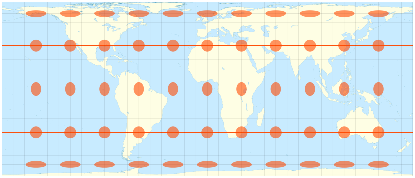
July 1, 2026

Surfaces: extrinsic vs. intrinsic

Extrinsic

Intrinsic



Four images from Wikipedia

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	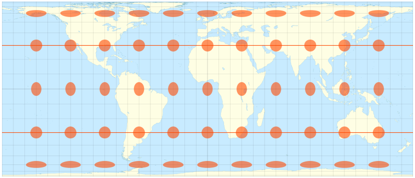
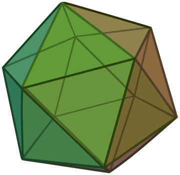	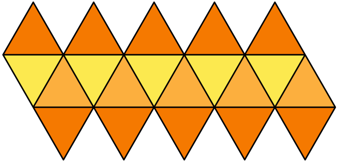

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	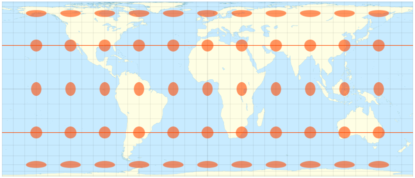
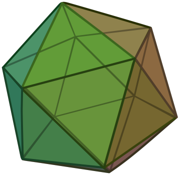	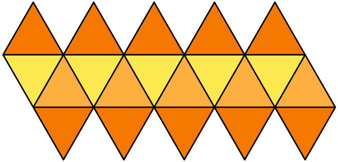

Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

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Discretizing continuous tangent vector fields on surfaces is “impossible”

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3. Continuous on each triangle.

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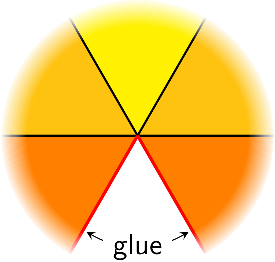

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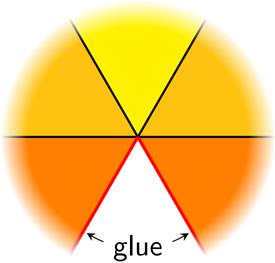

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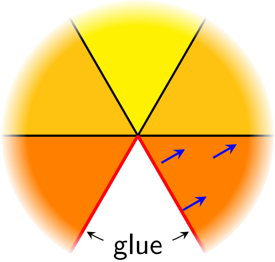

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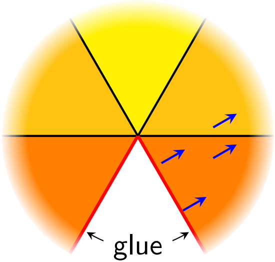

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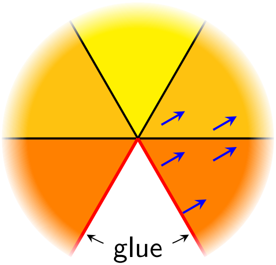

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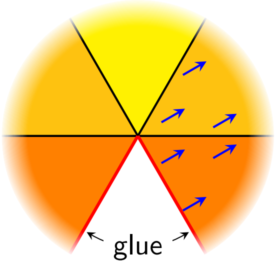

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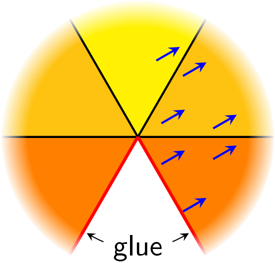

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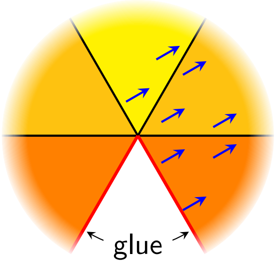

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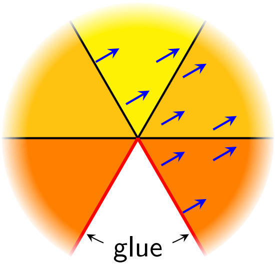

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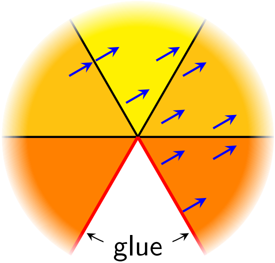

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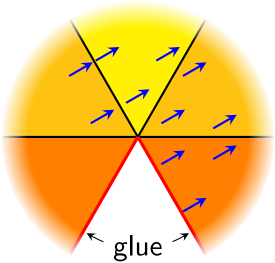

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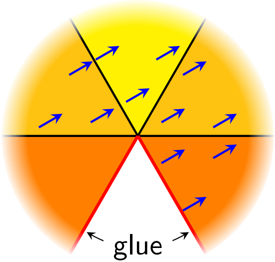

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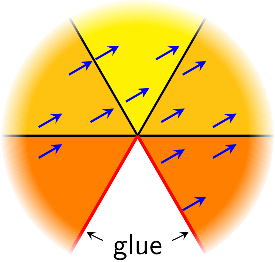

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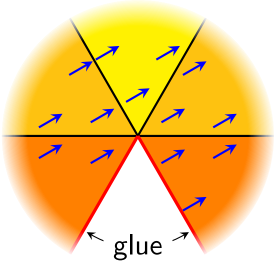

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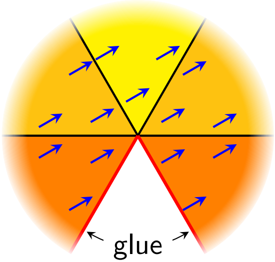

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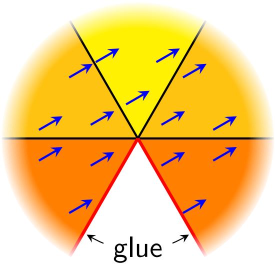

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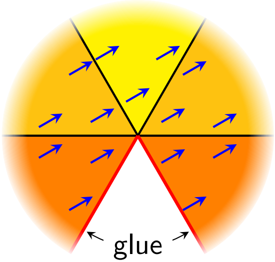

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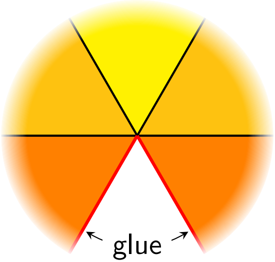

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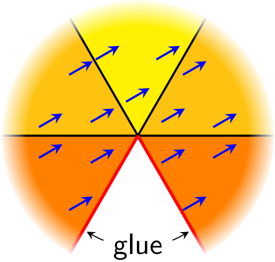

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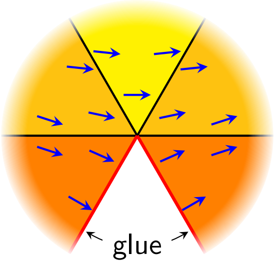

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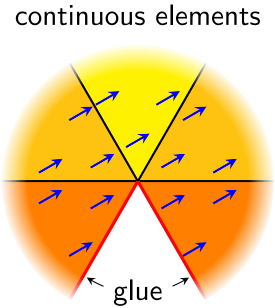

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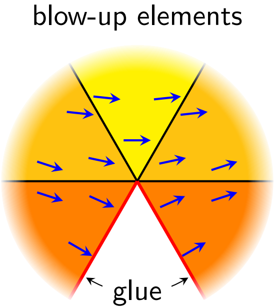

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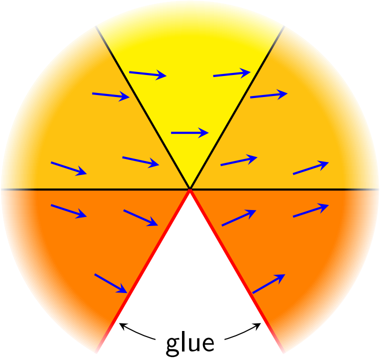
    - Intrinsic.
    - Discontinuities are *only* where the problem is (vertices with angle defect).

Break continuity at vertices

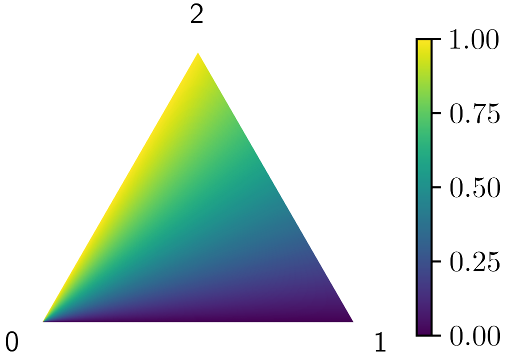
blow-up elements



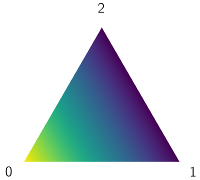
continuous across all edges

discontinuous at vertices

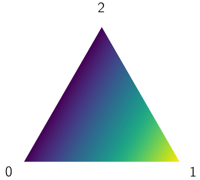
A simplicial analogue of the angular coordinate



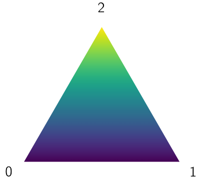
$$\frac{\lambda_2}{\lambda_1 + \lambda_2}$$



λ_0

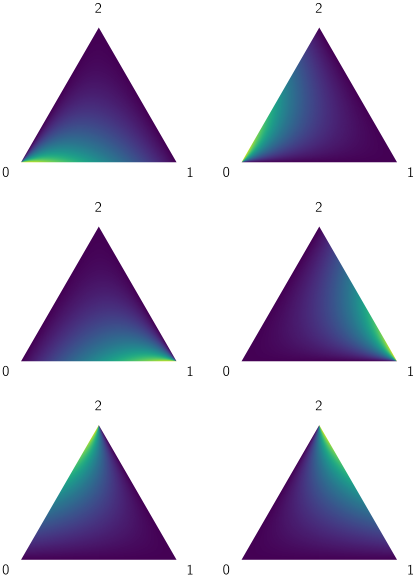


λ_1



λ_2

Blow-up $b\mathcal{P}_1$ shape functions



$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

New?

Shape function

$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2} = \frac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_2}{\lambda_2}.$$

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Earlier appearances

New?

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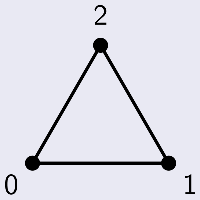
- Let t_0, t_1, t_2 be the times when the respective radiation sources produce their first particle.
- ψ_{012} is the probability that $t_0 \leq t_1 \leq t_2$.

Degrees of freedom

Classical Lagrange $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
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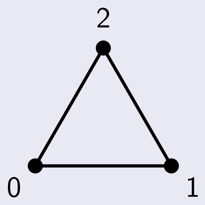

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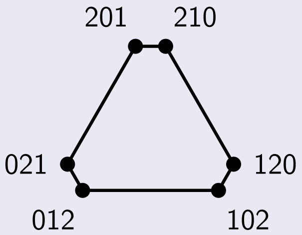
Blow-up $b\mathcal{P}_1$



- $012 : \lim\limits_{\lambda_1 \to 0} \lim\limits_{\lambda_2 \to 0}$
- $120 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_0 \to 0}$
- $201 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $021 : \lim\limits_{\lambda_2 \to 0} \lim\limits_{\lambda_1 \to 0}$
- $102 : \lim\limits_{\lambda_0 \to 0} \lim\limits_{\lambda_2 \to 0}$
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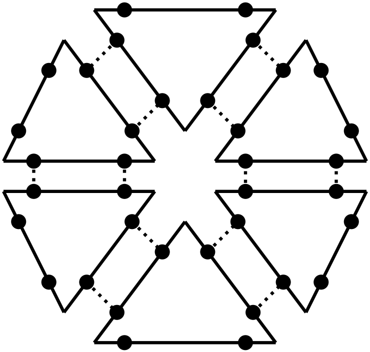
Global spaces



Blow-up finite elements

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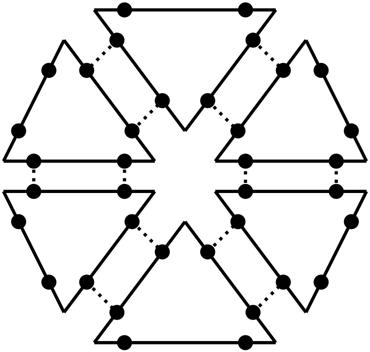
Global spaces



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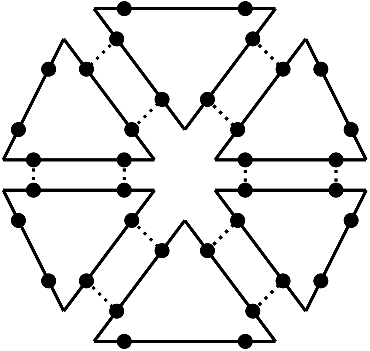

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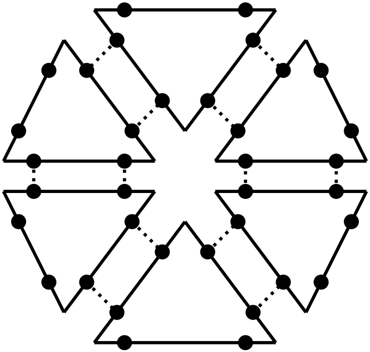

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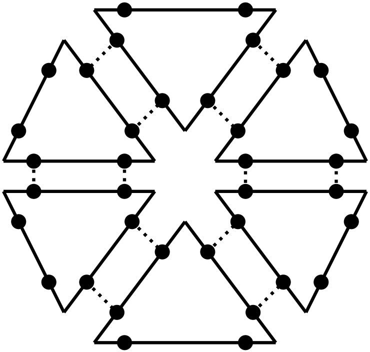

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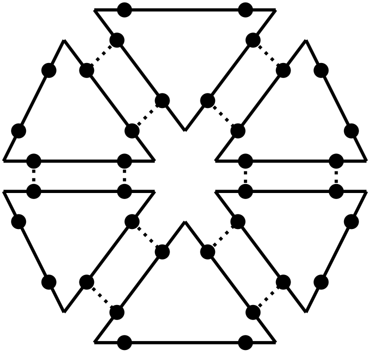

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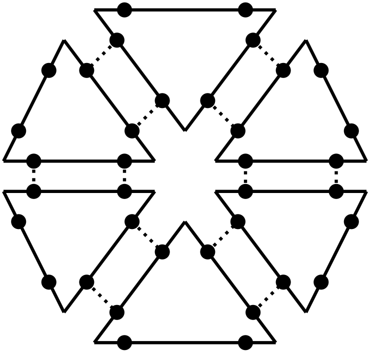

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- General tensor fields are analogous.

Vector Laplacian eigenvalue problems on surfaces

Hodge Laplacian (e.g. Maxwell)

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

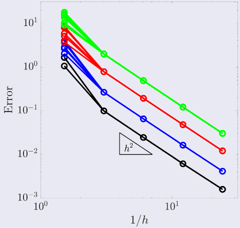
- Tangential continuity across edges suffices.
- Standard FEEC works.
- L^2 pairing suffices.

Bochner Laplacian (e.g. Stokes)

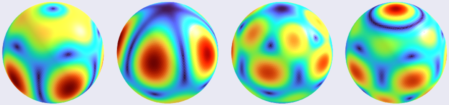
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

Bochner Laplacian on sphere using blow-up elements



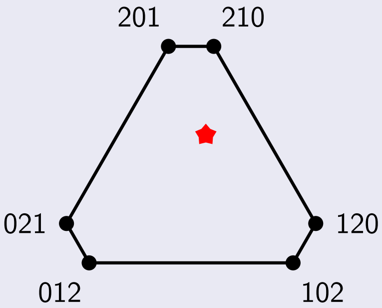
Eigenvalue error

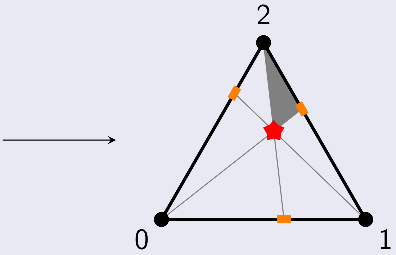


Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

Blowing up the simplex

Configuration space of barycentric subdivisions





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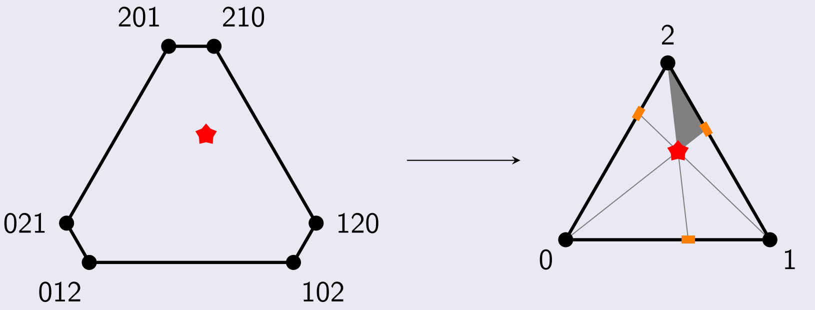

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Smooth $\longleftrightarrow$ Smooth "in polar coordinates"

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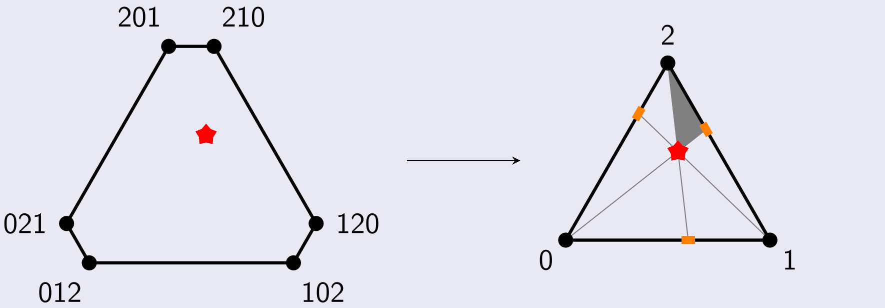

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- Another interpretation of $\psi_{012} = \lambda_0 \frac{\lambda_1}{\lambda_1+\lambda_2}$: the relative area of the shaded region.
- For the general theory of blowing up manifolds with corners, see Melrose, 1996.

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- Degrees of freedom of blow-up Whitney k-forms are integration over k-faces of the blow-up simplex.

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$$\int_M \langle R_{\mathrm{dist}} v, w\rangle \wedge \beta := \int_M \langle \nabla v \wedge \nabla w\rangle \wedge \beta + \int_M \langle \nabla v, w\rangle \wedge d\beta$$

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik
Blow-up Whitney forms, shadow forms, and Poisson processes.
Results in Applied Mathematics, special issue on Hilbert complexes, Paper No. 100529, 2025.

J. P. Brasselet, M. Goresky, and R. MacPherson.
Simplicial differential forms with poles.
Amer. J. Math., 113(6):1019–1052, 1991.

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