Finite element spaces for tensor fields

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Tangential and normal continuity of vector fields

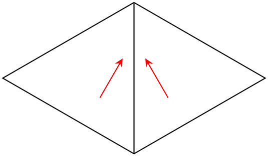


Figure: Tangential continuity (left) vs. normal continuity (right)

Tangential continuity

- Well-defined line integrals.
- In $H(\mathrm{curl})$.

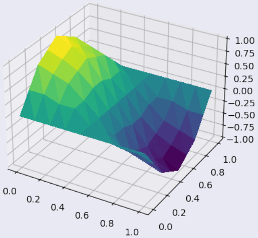
Normal continuity

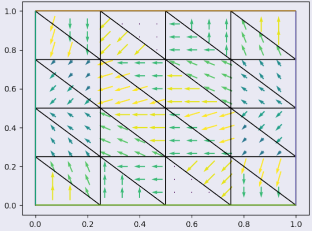
- Well-defined fluxes.
- In $H(\mathrm{div})$.

What's wrong with full continuity?

Finite element exterior calculus (FEEC) perspective: differential complexes

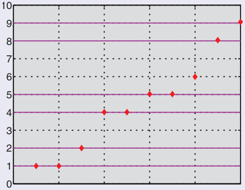
Gradients of scalar fields only have tangential continuity



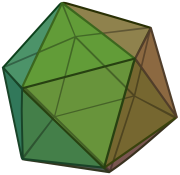


Spurious eigenvalues of the curl curl operator (AFW, 2010)

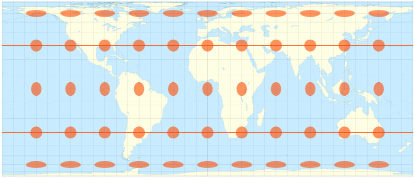
- Solve $\operatorname{curl}\operatorname{curl} u = \lambda u$, where u is a vector field on a square domain with appropriate boundary conditions.
- Using vector fields with full continuity yields false eigenvalue $\lambda = 6$.

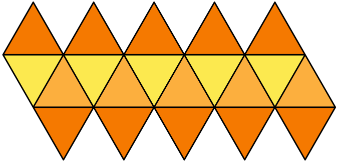


Extrinsic



Intrinsic





Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

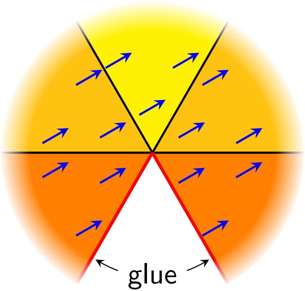
Four images from Wikipedia

What’s wrong with full continuity?

Geometric perspective: Angle defect obstruction to continuous elements

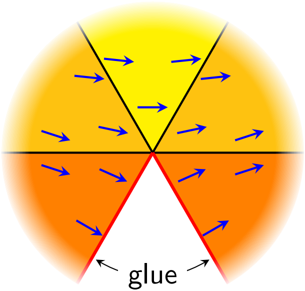
- Try to construct a tangent vector field on the icosahedron.
- What do we see when we zoom in on a vertex?

continuous elements



continuous on each triangle
discontinuous across red edge

blow-up elements



continuous across all edges
discontinuous at vertices

Metric-dependence vs. affine-invariance

Metric-dependent finite element spaces

- Defining finite element spaces of vector fields with full continuity requires a Riemannian metric (even via differential form proxies).
- Behavior depends on whether angle defect is zero or not.

Affine-invariant (metric-independent) finite element spaces

- FEEC differential forms Λ^k and their continuity conditions are defined without reference to a Riemannian metric.
- Same for double forms $\Lambda^{p,q}$.
- Angle defect cannot pose a problem since angle defect is not even defined without a Riemannian metric.
- In particular, for vector fields with tangential or normal continuity, FEEC works just as well on surface meshes as it does on the plane.

Section 1

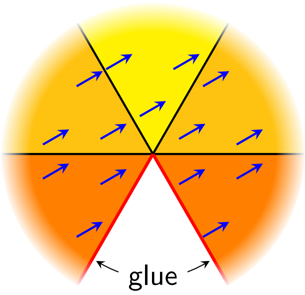
Metric-dependent finite element spaces: Blow-up elements

Metric-dependent finite element spaces

Motivating problem

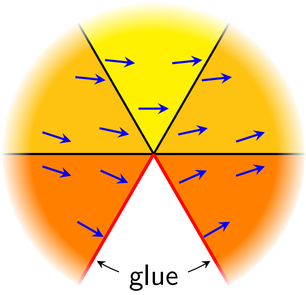
- Goal: construct intrinsic discretizations of tangent vector fields on smooth surfaces that are continuous across edges.
- Obstruction to using classical Lagrange $\mathcal{P}_1$ elements: angle defect.

continuous elements



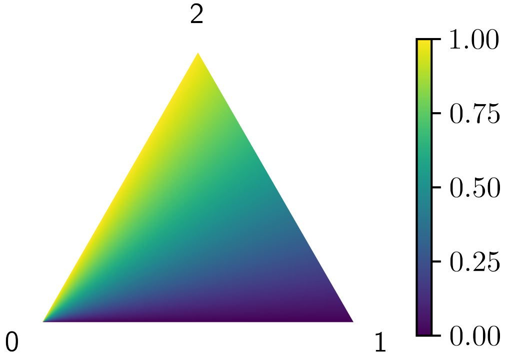
continuous on each triangle
discontinuous across red edge

blow-up elements



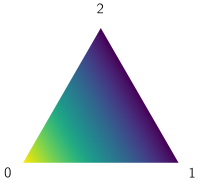
continuous across all edges
discontinuous at vertices

A simplicial analogue of the angular coordinate

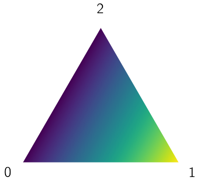


$$\frac{\lambda_2}{\lambda_1 + \lambda_2}$$

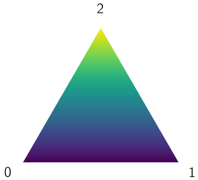
Lagrange $\mathcal{P}_1$ shape functions



λ_0

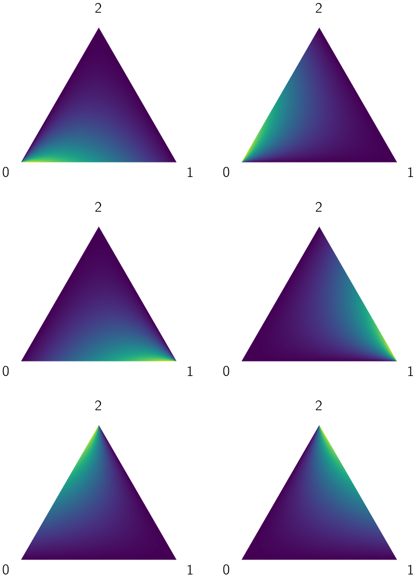


λ_1



λ_2

Blow-up $b\mathcal{P}_1$ shape functions



$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

New?

Shape function

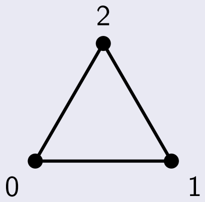
$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2} = \frac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_2}{\lambda_2}.$$

Earlier appearances

- Geometric invariants (Chen, 1957).
- Horse betting (Harville, 1973).
- Intersection homology (Brasselet, Goresky, MacPherson, 1991; Bendiffalah, 1995).

Degrees of freedom

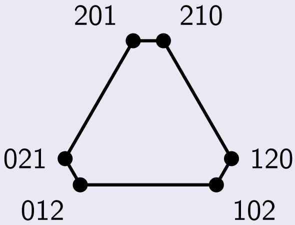
Classical Lagrange $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

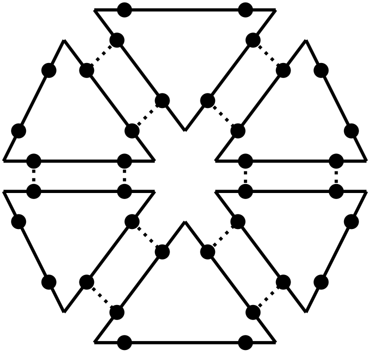
- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
- $2 : \lambda_2 = 1 \Leftrightarrow \lambda_0 = \lambda_1 = 0$

Blow-up $b\mathcal{P}_1$



- $012 : \lim\limits_{\lambda_1\to 0} \lim\limits_{\lambda_2\to 0}$
- $120 : \lim\limits_{\lambda_2\to 0} \lim\limits_{\lambda_0\to 0}$
- $201 : \lim\limits_{\lambda_0\to 0} \lim\limits_{\lambda_1\to 0}$
- $021 : \lim\limits_{\lambda_2\to 0} \lim\limits_{\lambda_1\to 0}$
- $102 : \lim\limits_{\lambda_0\to 0} \lim\limits_{\lambda_2\to 0}$
- $210 : \lim\limits_{\lambda_1\to 0} \lim\limits_{\lambda_0\to 0}$

Global spaces



Blow-up finite elements

- Scalar fields: we place a number at each dot.
- Vector fields: we place two numbers at each dot, for the tangential and normal components, respectively.
  - Enforce continuity for both components, yielding full continuity across edges.
- Matrix fields: At each dot, we record the tangential–tangential component, the tangential–normal component, etc.
  - Can impose conditions on the components such as symmetry, trace-free, etc.
  - Can enforce continuity for all components or just some of them.
- General tensor fields are analogous.

Vector Laplacian eigenvalue problems on surfaces

Hodge Laplacian (e.g. Maxwell)

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

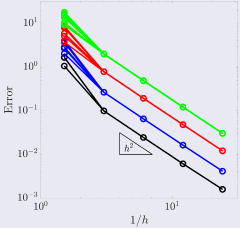
- Tangential continuity across edges suffices.
- Standard FEEC works.
- L^2 pairing suffices.

Bochner Laplacian (e.g. Stokes)

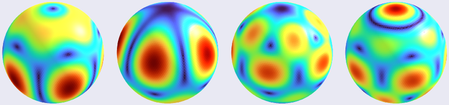
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

Bochner Laplacian on sphere using blow-up elements



Eigenvalue error



Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

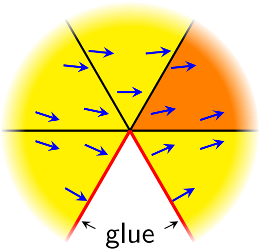
There's more

So far in this talk

- Lowest order blow-up elements in two dimensions, $b\mathcal{P}_1(T^2)$,
  - including tensor fields with components in $b\mathcal{P}_1(T^2)$.

Our paper

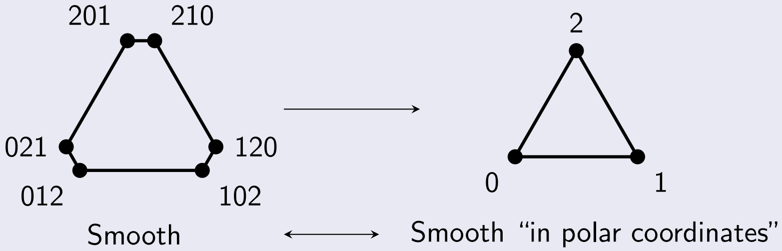
- Differential complex of blow-up Whitney forms in any dimension, $b\mathcal{P}_1^-\Lambda^k(T^n)$.
  - Shape functions previously studied in (Brasselet, Goresky, MacPherson, 1991), called shadow forms.
- Higher-order blow-up scalar fields $b\mathcal{P}_r(T^n)$.
- A surprising connection to arrival times of Poisson processes, yielding simpler computations.
  - Three radiation sources with rates λ_0, λ_1, and λ_2, sum 1.
  - Let t_0, t_1, t_2 be the times when the respective radiation sources produce their first particle.
  - $\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}$ is the probability that $t_0 \leq t_1 \leq t_2$.
- Degrees of freedom in terms of blow-up simplex.



- Even on an individual triangle, the vector field is not continuous at the origin.
- But it is "continuous in polar coordinates," i.e. in r and θ.

Blowing up manifolds with corners (Melrose, 1996)

- formalizes continuity/smoothness "in polar coordinates"



Section 2

Affine-invariant (metric-independent) finite element spaces: double forms

Differential forms corresponding to vector field $\langle M, N, P\rangle$

One-forms Λ^1

- $M\,dx + N\,dy + P\,dz$
- Restricted to the xy-plane $z = 0$:
  - $M\,dx + N\,dy$.
  - Tangential components.

Two-forms Λ^2

- $M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$.
- Restricted to the xy-plane $z = 0$:
  - $P\,dx \wedge dy$.
  - Normal component.

Continuity conditions

- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are $(n-1)$-forms.

Extending FEEC to matrices and tensors

Continuity conditions for 2-tensors (matrix fields)

- tangential–tangential
- normal–normal
- normal–tangential

Applications

- Strain/stress tensors
  - Elasticity (objects deforming under stress)
  - Fluid mechanics (Stokes equations)
- Numerical geometry/relativity
  - Riemannian/Minkowski metric
  - Curvature tensor

Double forms

Vector fields ($\mathbb{R}^3$)

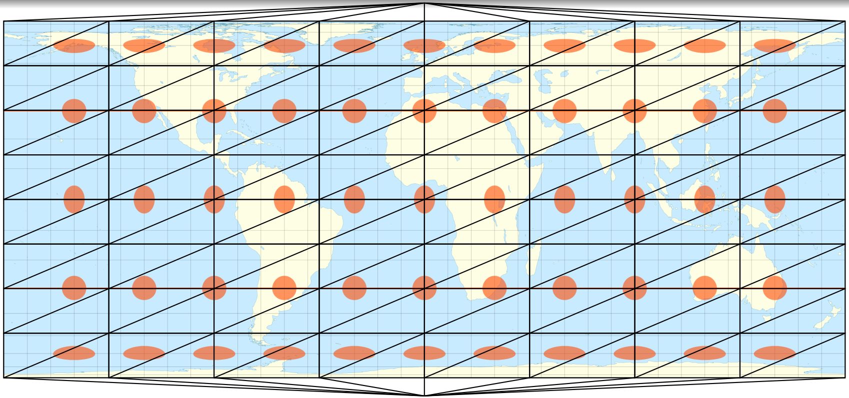
- Vector fields with tangential continuity are one-forms Λ^1.
- Vector fields with normal continuity are two-forms Λ^2.

Matrix fields ($\mathbb{R}^3 \otimes \mathbb{R}^3$)

- Matrix fields with tangential–tangential continuity are $(1,1)$-forms $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$.
- Matrix fields with normal–tangential continuity are $(2,1)$-forms $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$.
- Matrix fields with normal–normal continuity are $(2,2)$-forms $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$.

Regge metrics $\Lambda_0^{1,1}$

Symmetric matrix fields with tangential–tangential continuity



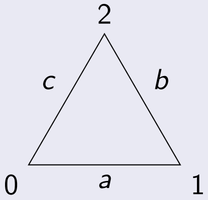
Regge finite elements

- Record the length of each edge.
- For each triangle, use the corresponding Euclidean metric.
- Get piecewise constant metric with tang.–tang. continuity.

Map credit: Wikipedia, Gaba

Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



Regge metric:

$$\begin{aligned} g = &-\tfrac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0) \\ &-\tfrac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) \\ &-\tfrac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2) \end{aligned}$$

Observations

- If $\mathbf{v}$ is the vector from vertex 0 to vertex 1, then

  $$d\lambda_0(\mathbf{v}) = -1, \qquad d\lambda_1(\mathbf{v}) = 1, \qquad d\lambda_2(\mathbf{v}) = 0.$$

  As desired:

  $$g(\mathbf{v}, \mathbf{v}) = -\tfrac{1}{2}a^2(-1-1) - \tfrac{1}{2}b^2(0+0) - \tfrac{1}{2}c^2(0+0) = a^2.$$

- Crucial: $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ is zero on other edges.

Constant coefficient finite elements for bilinear forms

Geometrically decomposed bases for finite element spaces

- Each basis element φ must be associated to a face F of the triangulation, such that, for any other face G,

$$\varphi \text{ is nonzero on } G \Leftrightarrow G \geq F.$$

Constant coefficient symmetric bilinear forms $\Lambda^{1,1}_{\mathrm{sym}}$

- Regge's construction works in any dimension. To each edge ij, associate

$$d\lambda_i \odot d\lambda_j := d\lambda_i \otimes d\lambda_j + d\lambda_j \otimes d\lambda_i.$$

Constant coefficient antisymmetric bilinear forms $\Lambda^{1,1}_{\mathrm{asym}}$

- Finite element spaces do not exist in dimension ≥ 3.
- In 3D, antisymmetric bilinear forms $\leftrightarrow$ vector fields with normal continuity.
- A nonzero constant vector field can't be tangent to three faces of a tetrahedron.

Affine-invariant subspaces of double forms

Theorem (Eigendecomposition of s^*s)

$$\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m, \qquad \max\{0, q-p\} \leq m \leq \min\{q, n-p\}.$$

Example

- $\Lambda^{1,1}_0$: Symmetric bilinear forms, $\varphi(X; Y) = \varphi(Y; X)$.
- $\Lambda^{1,1}_1$: Λ^2, antisymmetric bilinear forms, $\varphi(X; Y) = -\varphi(Y; X)$.

- $\Lambda^{2,1}_0$: spanned by $\alpha \otimes \beta$ such that $\alpha \wedge \beta = 0$.
  - Matrix proxy in 3D: trace-free matrices.
- $\Lambda^{2,1}_1$: Λ^3.
  - Matrix proxy in 3D: multiples of the identity matrix.

- $\Lambda^{2,2}_0$: Symmetric, satisfying the algebraic Bianchi identity.
  - Riemann curvature tensor.
- $\Lambda^{2,2}_1$: Antisymmetric, $\varphi(X, Y; Z, W) = -\varphi(Z, W; X, Y)$.
- $\Lambda^{2,2}_2$: Λ^4.

Finite element spaces

Theorem (—, Gawlik)

Apart from $\Lambda^{p,q}_q \cong \Lambda^{p+q}$ with constant coefficients, there is a finite element space for every natural space of double forms $\Lambda^{p,q}_m$ with polynomial coefficients of any degree (including zero).

Example (Constant coefficient spaces)

- $\Lambda^{1,1}_0$: symmetric matrices with tangential–tangential continuity (Regge, 1961; Christiansen, 2004).
  - Higher order: (Li, 2018).
- $\Lambda^{2,1}_0$ in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda^{2,2}_0$ in 3D: symmetric matrices with normal–normal continuity (Pechstein and Schöberl, 2011).
- $\Lambda^{2,2}_0$ (or $\Lambda^{n-2,n-2}_0$) in any dimension: finite elements for the Riemann curvature tensor.

Degrees of freedom for constant coefficient spaces

				d			
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	**1**	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	**2**	0	0	0	0
$\Lambda_0^{2,2}$	0	0	**1**	**2**	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	**3**	0	0	0
$\Lambda_0^{3,2}$	0	0	0	**3**	**5**	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	**4**	0	0
$\Lambda_0^{3,3}$	0	0	0	**1**	**5**	**5**	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	**6**	**9**	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	**5**	0

Table: Number of degrees of freedom for $\Lambda_m^{p,q}$ associated to a face of the triangulation of dimension d is $\frac{p-q+2m+1}{p+m+1}\binom{d+1}{q-m}\binom{q-m-1}{d-p-m}$.

Section 3

More on $\mathcal{P}_r\Lambda_0^{2,2}$ (Joint with Lily DiPaulo)

Algebraic curvature tensors $\Lambda_0^{2,2}$

The space $\Lambda_0^{2,2}$

- Symmetric $(2,2)$-forms satisfying the Bianchi identity.
- $\Lambda_0^{2,2}$ is spanned by $\alpha \odot \beta$ where $\alpha, \beta \in \Lambda^2$ and $\alpha \wedge \beta = 0$.

Finite element spaces

- Construct bases for constant coefficient spaces using (—, Gawlik)
- Generalize to higher order similarly to Li's work on Regge finite elements.

Regge finite elements $\mathcal{P}_r\Lambda_0^{1,1}$ (symmetric bilinear forms)

Constant coefficient space $\Lambda_0^{1,1}$

- For i and j distinct vertices, associate $d\lambda_i \odot d\lambda_j$ to edge ij.
- These forms are a basis for the space $\Lambda_0^{1,1}$ of symmetric bilinear forms with constant coefficients.

Higher order spaces $\mathcal{P}_r\Lambda_0^{1,1}$ (Li)

- For a multiindex I, let λ^I be the corresponding monomial, and let $\operatorname{supp} I$ denote the set of vertices whose corresponding exponent is at least one in λ^I.
  - e.g. if $\lambda^I = \lambda_0^5\lambda_3^4$ then $\operatorname{supp} I = \{0, 3\}$.
- Associate $\lambda^I\, d\lambda_i \odot d\lambda_j$ to the face with vertices $\{i, j\} \cup \operatorname{supp} I$.
- These forms are a basis for $\mathcal{P}_r\Lambda_0^{1,1}$ because the monomials are a basis for $\mathcal{P}_r$ and the $d\lambda_i \odot d\lambda_j$ are a basis for $\Lambda_0^{1,1}$.

Algebraic curvature tensors $\Lambda_0^{2,2}$

Constant coefficient space $\Lambda_0^{2,2}$

- Let $d\lambda_{ij} := d\lambda_i \wedge d\lambda_j$.
- To each two-dimensional face ijk, associate

$$\beta_{ijk} := d\lambda_{ij} \odot d\lambda_{jk} + d\lambda_{jk} \odot d\lambda_{ki} + d\lambda_{ki} \odot d\lambda_{ij}$$

- To each three-dimensional face $ijkl$, associate

$$\begin{aligned} \gamma_{iklj} &:= d\lambda_{il} \odot d\lambda_{jk} - d\lambda_{ij} \odot d\lambda_{kl}, \\ \gamma_{iljk} &:= d\lambda_{ij} \odot d\lambda_{kl} - d\lambda_{ik} \odot d\lambda_{lj}. \end{aligned}$$

- These forms are a basis for the space $\Lambda_0^{2,2}$ of algebraic curvature tensors with constant coefficients.
- These formulas can be derived from the representation theory of the symmetric group (Young diagrams), following (—, Gawlik).

Higher order algebraic curvature tensors $\mathcal{P}_r\Lambda_0^{2,2}$

Constant coefficient space $\Lambda_0^{2,2}$

$$\begin{aligned}
\beta_{ijk} &:= d\lambda_{ij} \odot d\lambda_{jk} + d\lambda_{jk} \odot d\lambda_{ki} + d\lambda_{ki} \odot d\lambda_{ij}, \\
\gamma_{iklj} &:= d\lambda_{il} \odot d\lambda_{jk} - d\lambda_{ij} \odot d\lambda_{kl}, \\
\gamma_{iljk} &:= d\lambda_{ij} \odot d\lambda_{kl} - d\lambda_{ik} \odot d\lambda_{lj}.
\end{aligned}$$

Higher order space $\mathcal{P}_r\Lambda_0^{2,2}$

- Associate $\lambda^I\beta_{ijk}$ to the face with vertices $\{i, j, k\} \cup \operatorname{supp} I$.
- Associate $\lambda^I\gamma_{iklj}$ and $\lambda^I\gamma_{iljk}$ to the face with vertices $\{i, j, k, l\} \cup \operatorname{supp} I$.
- These forms are a geometrically decomposed basis for $\mathcal{P}_r\Lambda_0^{2,2}$.

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik
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Finite element spaces of double forms.
https://arxiv.org/abs/2505.17243

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Finite element spaces of double two-forms with polynomial coefficients.
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Sequences from sequences, sans coordinates.
In preparation.

Supported by NSF DMS-2411209.