Two approaches for discretizing spaces of tensors with specified interelement continuity conditions

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Outline

Section 1

Introduction: Continuity conditions

Tangential and normal continuity of vector fields

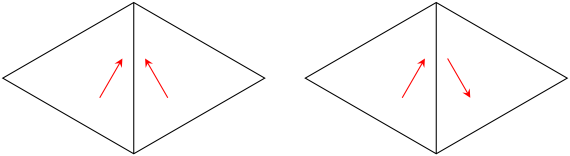


Figure: Tangential continuity (left) vs. normal continuity (right)

Tangential continuity

- Well-defined line integrals.
- In $H(\mathrm{curl})$.

Normal continuity

- Well-defined fluxes.
- In $H(\mathrm{div})$.

Differential forms corresponding to vector field $\langle M, N, P\rangle$

One-forms Λ^1

- $M\,dx + N\,dy + P\,dz$
- Restricted to the xy-plane $z = 0$:
  - $M\,dx + N\,dy$.
  - Tangential components.

Two-forms Λ^2

- $M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$.
- Restricted to the xy-plane $z = 0$:
  - $P\,dx \wedge dy$.
  - Normal component.

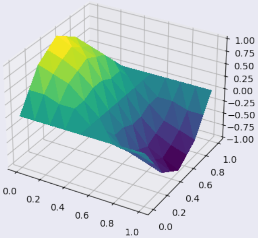
Continuity conditions

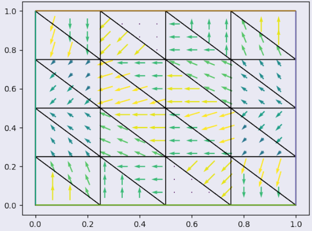
- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are $(n-1)$-forms.

What's wrong with full continuity?

FEEC perspective: differential complexes

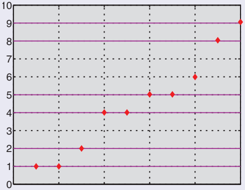
Gradients of scalar fields only have tangential continuity





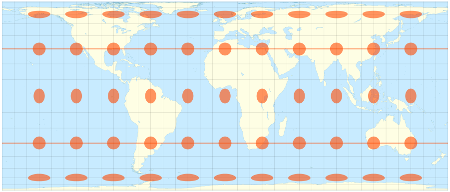
Spurious eigenvalues of the curl curl operator (AFW, 2010)

- Solve $\operatorname{curl}\operatorname{curl} u = \lambda u$, where u is a vector field on a square domain with appropriate boundary conditions.
- Using vector fields with full continuity yields false eigenvalue $\lambda = 6$.



What's wrong with full continuity?

Geometric perspective



Four images from Wikipedia

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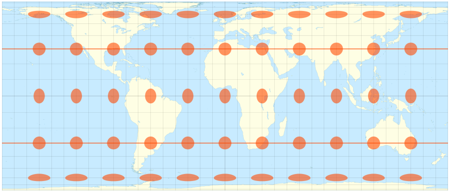

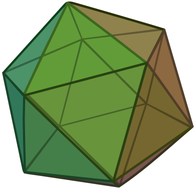

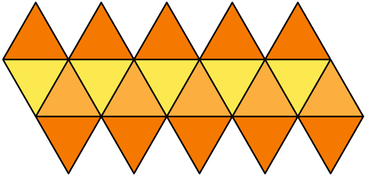

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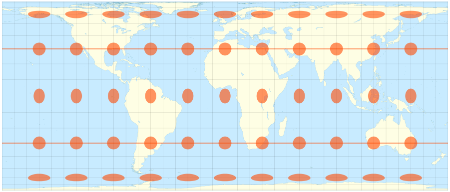
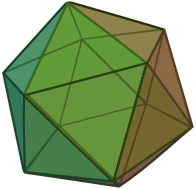
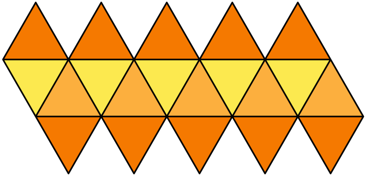

Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

Four images from Wikipedia

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Geometric perspective: Angle defect obstruction to continuous elements

- Try to construct a tangent vector field on the icosahedron.

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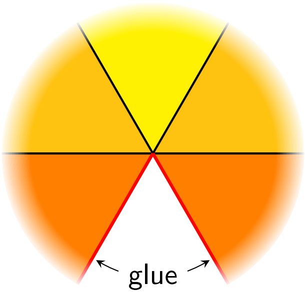

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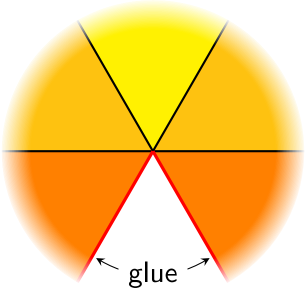

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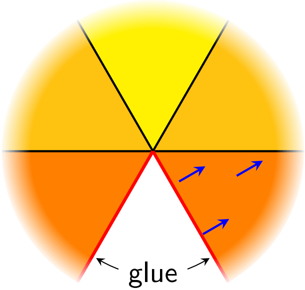

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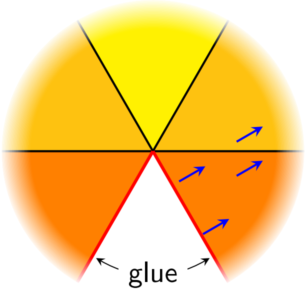

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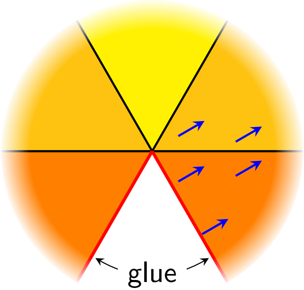

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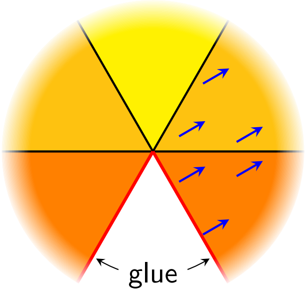

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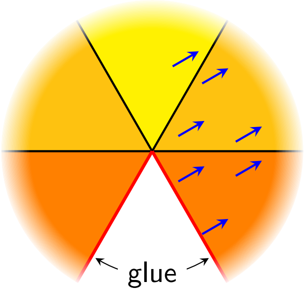

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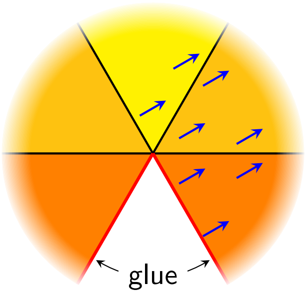

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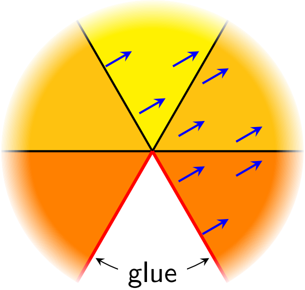

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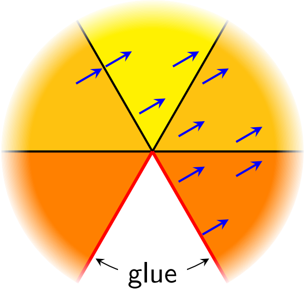

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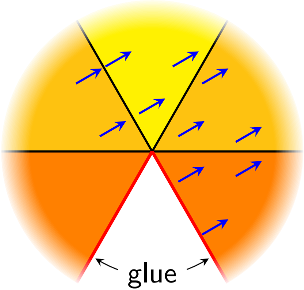

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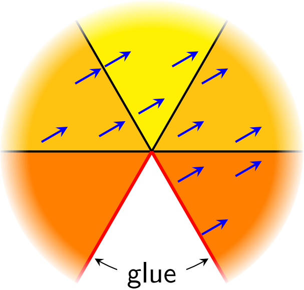

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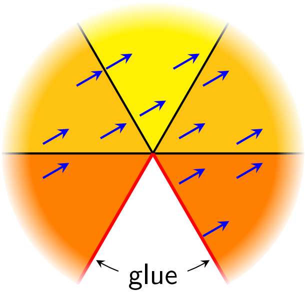

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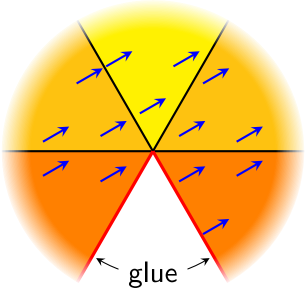

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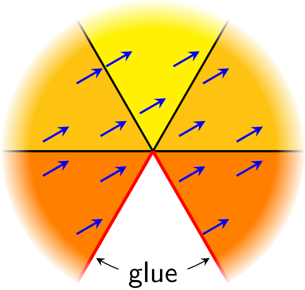

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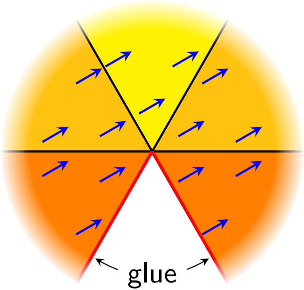

continuous on each triangle

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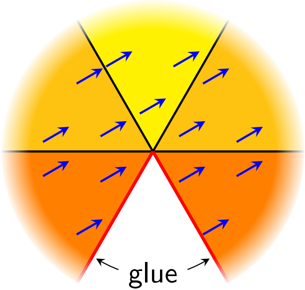

continuous on each triangle
discontinuous across red edge

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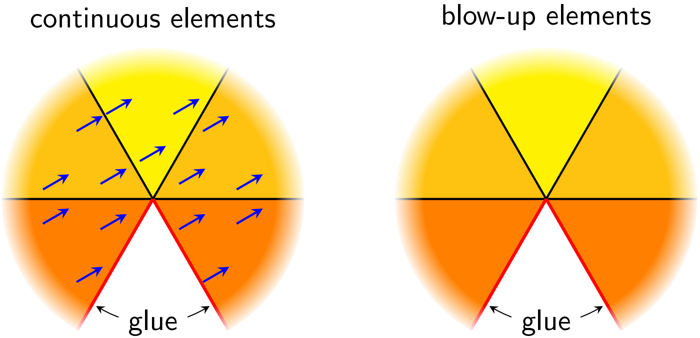

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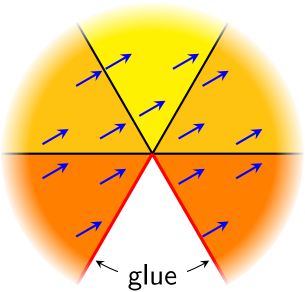

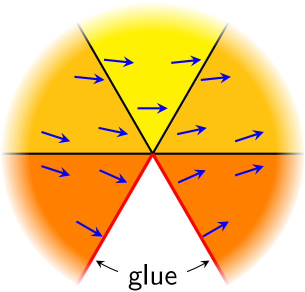

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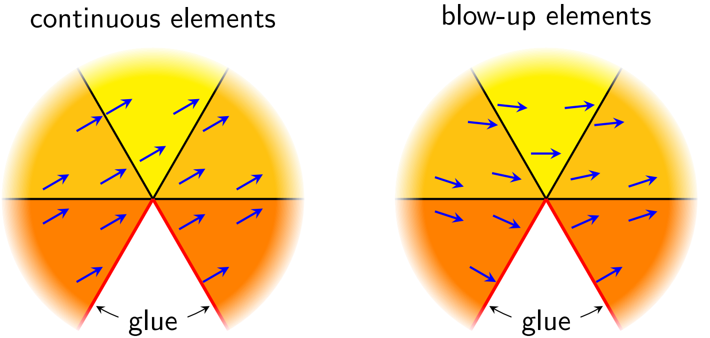

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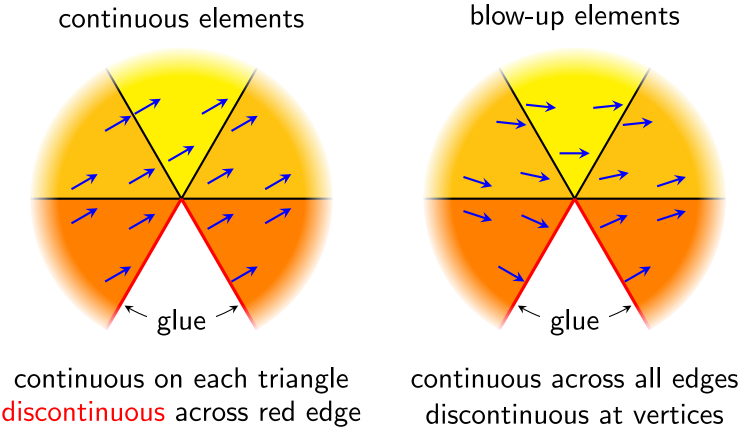


Section 2

Double forms: Matrix fields with tangential or normal continuity, Riemann curvature tensor

Matrix fields and tensor fields

Continuity conditions for matrix fields

- tangential–tangential
- normal–normal
- normal–tangential

Applications

- Strain/stress tensors
  - Elasticity (objects deforming under stress)
  - Fluid mechanics (Stokes equations)
- Curvature tensor
  - Numerical geometry
  - Numerical relativity

Double forms

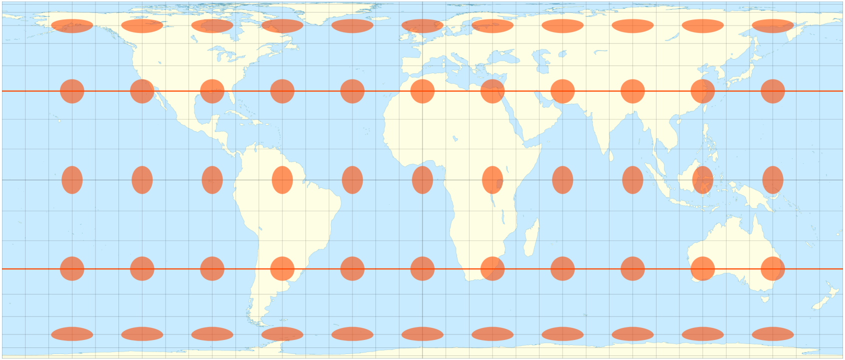
Vector fields ($\mathbb{R}^3$)

- Vector fields with tangential continuity are one-forms Λ^1.
- Vector fields with normal continuity are two-forms Λ^2.

Matrix fields ($\mathbb{R}^3 \otimes \mathbb{R}^3$)

- Matrix fields with tangential–tangential continuity are $(1,1)$-forms $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$.
- Matrix fields with normal–tangential continuity are $(2,1)$-forms $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$.
- Matrix fields with normal–normal continuity are $(2,2)$-forms $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$.

Intrinsic geometry with Regge metrics



Map credit: Wikipedia, Gaba

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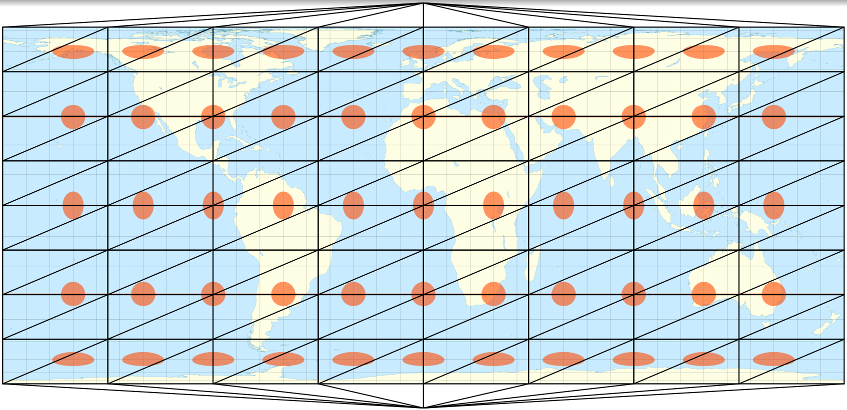

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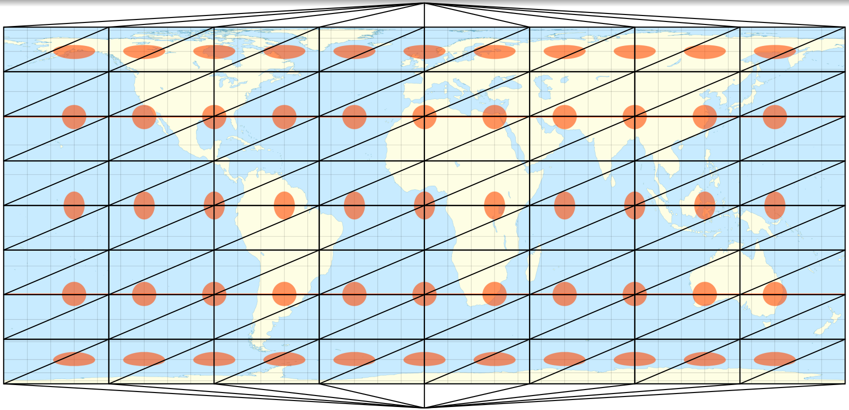


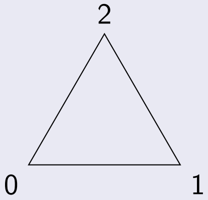
Regge finite elements

- Record the length of each edge.
- For each triangle, use the corresponding Euclidean metric.
- Get piecewise constant metric with tang.–tang. continuity.

Map credit: Wikipedia, Gaba

Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



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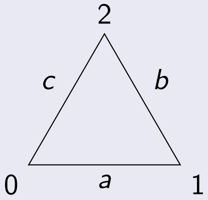

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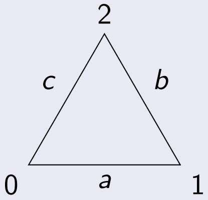

Regge metric:

$$g = -\tfrac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0) - \tfrac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) - \tfrac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2)$$

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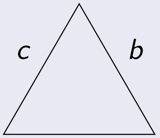

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Observations

- If $\mathbf{v}$ is the vector from vertex 0 to vertex 1, then

$$d\lambda_0(\mathbf{v}) = -1, \qquad d\lambda_1(\mathbf{v}) = 1, \qquad d\lambda_2(\mathbf{v}) = 0.$$

As desired:

$$g(\mathbf{v}, \mathbf{v}) = -\tfrac{1}{2}a^2(-1-1) - \tfrac{1}{2}b^2(0+0) - \tfrac{1}{2}c^2(0+0) = a^2.$$

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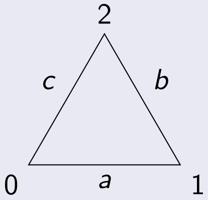

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- Crucial: $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ is zero on other edges.

Constant coefficient finite elements for bilinear forms

Local bases for finite element spaces

- Each basis element φ must be associated to a face F of the triangulation, such that, for any other face G,

$$\varphi \text{ is nonzero on } G \Leftrightarrow G \geq F.$$

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Constant coefficient symmetric bilinear forms $\Lambda^{1,1}_{\text{sym}}$

- Regge's construction works in any dimension. To each edge ij, associate

$$d\lambda_i \otimes d\lambda_j + d\lambda_j \otimes d\lambda_i.$$

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Constant coefficient antisymmetric bilinear forms $\Lambda^{1,1}_{\mathrm{asym}}$

- Finite element spaces do not exist in dimension ≥ 3.
- In 3D, antisymmetric bilinear forms $\leftrightarrow$ vector fields with normal continuity.
- A nonzero constant vector field can't be tangent to three faces of a tetrahedron.

Natural subspaces of double forms

Theorem (Eigendecomposition of s^*s)

$$\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m, \qquad \max\{0, q-p\} \le m \le \min\{q, n-p\}.$$

Example

- $\Lambda^{1,1}_0$: Symmetric bilinear forms, $\varphi(X;Y) = \varphi(Y;X)$.
- $\Lambda^{1,1}_1$: Λ^2, antisymmetric bilinear forms, $\varphi(X;Y) = -\varphi(Y;X)$.

- $\Lambda^{2,1}_0$: spanned by $\alpha \otimes \beta$ such that $\alpha \wedge \beta = 0$.
  - Matrix proxy in 3D: trace-free matrices.
- $\Lambda^{2,1}_1$: Λ^3.
  - Matrix proxy in 3D: multiples of the identity matrix.

- $\Lambda^{2,2}_0$: Symmetric, satisfying the algebraic Bianchi identity.
  - Riemann curvature tensor.
- $\Lambda^{2,2}_1$: Antisymmetric, $\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$.
- $\Lambda^{2,2}_2$: Λ^4.

Finite element spaces

Theorem

Apart from $\Lambda^{p,q}_q \cong \Lambda^{p+q}$ with constant coefficients, there is a finite element space for every natural space of double forms $\Lambda^{p,q}_m$ with polynomial coefficients of any degree (including zero).

Example (Constant coefficient spaces)

- $\Lambda^{1,1}_0$: symmetric matrices with tangential–tangential continuity (Regge, 1961).
  - Higher order: (Li, 2018).
- $\Lambda^{2,1}_0$ in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda^{2,2}_0$ in 3D: symmetric matrices with normal–normal continuity (Pechstein and Schöberl, 2011).
- $\Lambda^{2,2}_0$ (or $\Lambda^{n-2,n-2}_0$) in any dimension: finite elements for the Riemann curvature tensor.

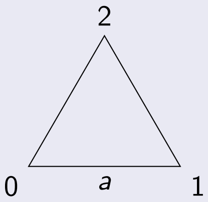
Degrees of freedom for constant coefficient spaces

	d						
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	**1**	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	**2**	0	0	0	0
$\Lambda_0^{2,2}$	0	0	**1**	**2**	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	**3**	0	0	0
$\Lambda_0^{3,2}$	0	0	0	**3**	**5**	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	**4**	0	0
$\Lambda_0^{3,3}$	0	0	0	**1**	**5**	**5**	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	**6**	**9**	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	**5**	0

Table: Number of degrees of freedom for $\Lambda_m^{p,q}$ associated to a face of the triangulation of dimension d is $\frac{p-q+2m+1}{p+m+1}\binom{d+1}{q-m}\binom{q-m-1}{d-p-m}$.

Extension

Recall



- It was crucial that $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ vanishes on the other edges.

Extension operators

- We need to be able to take a form on edge 01, and extend it to the triangle so that it vanishes on the other edges.
- The metric on edge 01 is $a^2\, d\lambda_1 \otimes d\lambda_1$.
- However, if we extend to the triangle using the formula $a^2\, d\lambda_1 \otimes d\lambda_1$, it won't vanish on edge 12.
- We first need to use $d\lambda_0 + d\lambda_1 = 0$ to rewrite $a^2\, d\lambda_1 \otimes d\lambda_1$ as $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ on edge 01.

Constructing extensions

Example ($\mathcal{P}_r\Lambda^{p,q}_m = \mathcal{P}_0\Lambda^{1,1}_0$)

1. Start with a form on edge 01 with vanishing trace: $d\lambda_1 \otimes d\lambda_1$
2. $\lambda_i = u_i^2, \quad d\lambda_i = 2u_i\,du_i$: $4u_1^2\,du_1 \otimes du_1$.
3. $u_0\,du_0 + u_1\,du_1$ wedge with each factor:
   $$4u_0^2u_1^2(du_0 \wedge du_1) \otimes (du_0 \wedge du_1).$$
4. Hodge star both factors (as forms on $\mathbb{R}^2$): $4u_0^2u_1^2$.
5. Divide by u_0u_1: $4u_0u_1$.
6. Divide by $(2r+p+m+1)(2r+q-m) = 2$: $2u_0u_1$.
7. Exterior derivative on both factors: $2(du_0 \otimes du_1 + du_1 \otimes du_0)$.
8. Apply $(-1)^{p+q}$ times the inverse Hodge star:
   $$-2(du_1 \otimes du_0 + du_0 \otimes du_1).$$
9. Multiply by u_0u_1: $-2u_0u_1(du_1 \otimes du_0 + du_0 \otimes du_1)$.
10. Convert back to λ_i: $-\frac{1}{2}(d\lambda_1 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_1)$.

Section 3

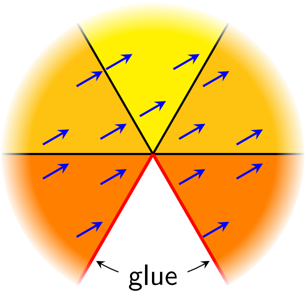
Blow-up finite elements: Any continuity conditions you like

Motivation

Motivating problem

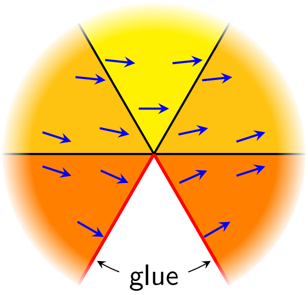
- Goal: construct intrinsic discretizations of tangent vector fields on smooth surfaces that are continuous across edges.
- Obstruction to using classical $\mathcal{P}_1$ elements: angle defect.

continuous elements



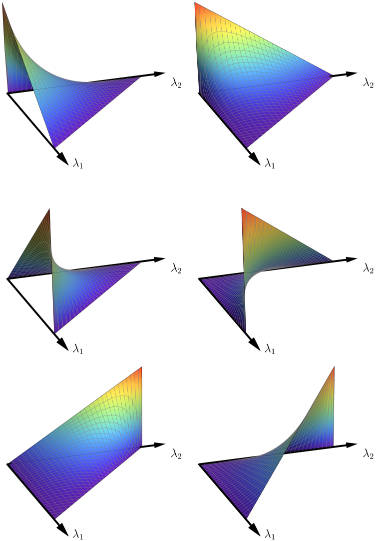
continuous on each triangle
discontinuous across red edge

blow-up elements



continuous across all edges
discontinuous at vertices

New finite element space



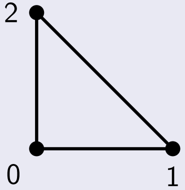
$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2+\lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0+\lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2+\lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0+\lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1+\lambda_0}.$$

Degrees of freedom

Classical $\mathcal{P}_1$

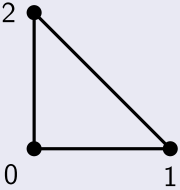


Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
- $2 : \lambda_2 = 1 \Leftrightarrow \lambda_0 = \lambda_1 = 0$

Degrees of freedom

Classical $\mathcal{P}_1$



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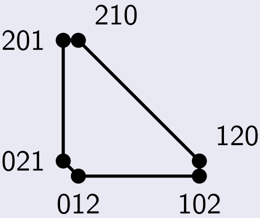
Blow-up $b\mathcal{P}_1$



- $012 : \lim_{\lambda_1 \to 0} \lim_{\lambda_2 \to 0}$
- $120 : \lim_{\lambda_2 \to 0} \lim_{\lambda_0 \to 0}$
- $201 : \lim_{\lambda_0 \to 0} \lim_{\lambda_1 \to 0}$
- $021 : \lim_{\lambda_2 \to 0} \lim_{\lambda_1 \to 0}$
- $102 : \lim_{\lambda_0 \to 0} \lim_{\lambda_2 \to 0}$
- $210 : \lim_{\lambda_1 \to 0} \lim_{\lambda_0 \to 0}$

Example: Evaluating degrees of freedom

Recall

$$\lambda_0 + \lambda_1 + \lambda_2 = 1, \qquad \psi_{012} = \frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2}.$$

Example: Evaluating degrees of freedom

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Evaluating degrees of freedom

$$012 : \lim_{\lambda_1\to 0}\lim_{\lambda_2\to 0} \tfrac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_1\to 0} \tfrac{\lambda_0\lambda_1}{\lambda_1} = \lim_{\lambda_0\to 1} \lambda_0 = 1,$$

$$021 : \lim_{\lambda_2\to 0}\lim_{\lambda_1\to 0} \tfrac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_2\to 0} \tfrac{0}{\lambda_2} = 0,$$

$$120 : \lim_{\lambda_2\to 0}\lim_{\lambda_0\to 0} \tfrac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_2\to 0} \tfrac{0}{1} = 0,$$

$$102 : \lim_{\lambda_0\to 0}\lim_{\lambda_2\to 0} \tfrac{\lambda_0\lambda_1}{\lambda_1+\lambda_2} = \lim_{\lambda_0\to 0} \tfrac{\lambda_0\lambda_1}{\lambda_1} = 0,$$

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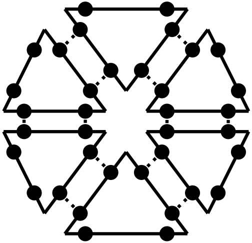
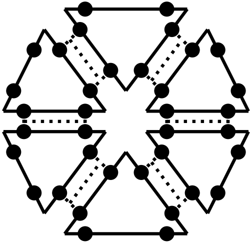
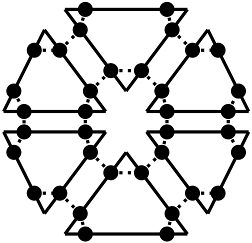
Global spaces



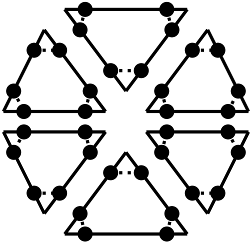
Blow-up finite elements



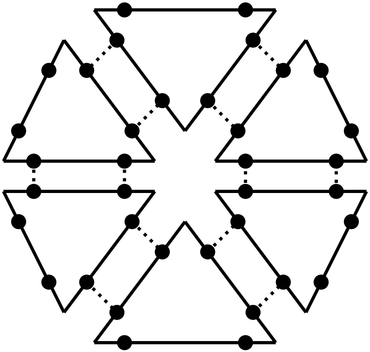
Crouzeix–Raviart–style blow-up elements



Lagrange



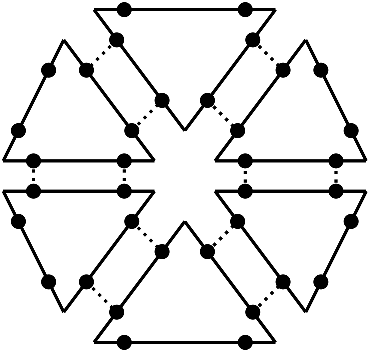
Discontinuous Lagrange



Blow-up finite elements

- Scalar fields: we placed a number at each dot.

Blow-up finite elements for tensors



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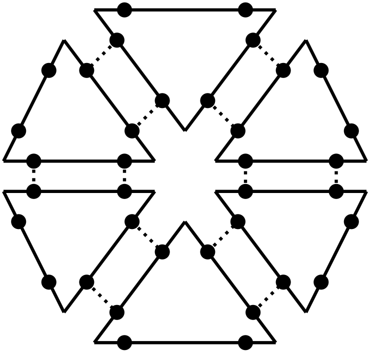

Blow-up finite elements

- Scalar fields: we placed a number at each dot.
- Vector fields: we place two numbers at each dot, for the tangential and normal components, respectively.
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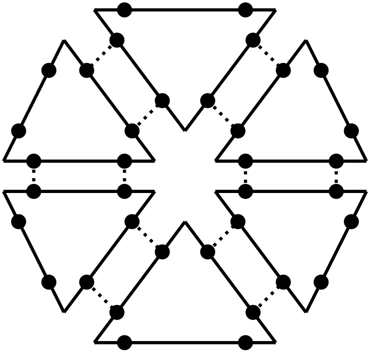
Blow-up finite elements for tensors



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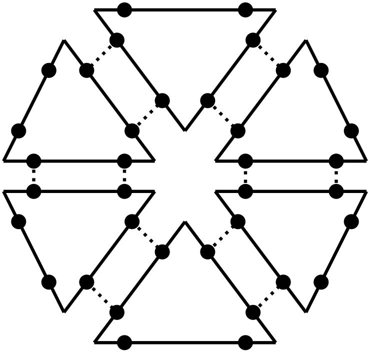

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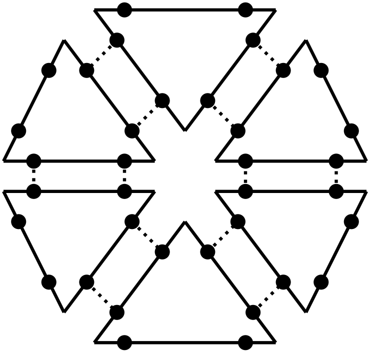

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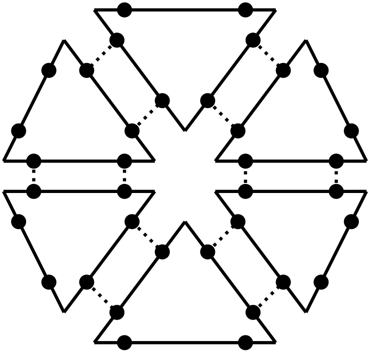

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  - Can enforce continuity for all components or just some of them.
- General tensor fields are analogous.

Vector Laplacian eigenvalue problems on surfaces

Hodge Laplacian

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

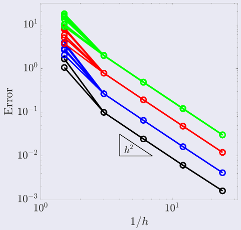
- Tangential continuity suffices.
- Standard FEEC works.
- L^2 pairing suffices.

Bochner Laplacian

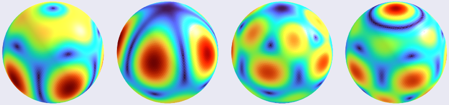
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

Bochner Laplacian on sphere using blow-up elements



Eigenvalue error



Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

There's more

This talk so far

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There's more

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  - Three radiation sources with rates λ_0, λ_1, and λ_2, sum 1.

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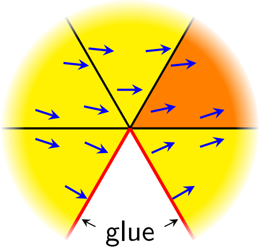
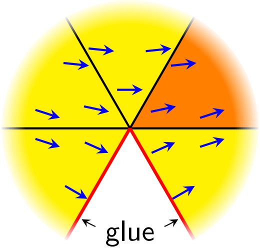
  - $\frac{\lambda_0 \lambda_1}{\lambda_1 + \lambda_2}$ is the probability that $t_0 \leq t_1 \leq t_2$.
- Degrees of freedom in terms of blow-up simplex.

Blowing up



- Even on an individual triangle, the vector field is not continuous at the origin.
- But it is "continuous in polar coordinates," i.e. in r and θ.

Blowing up



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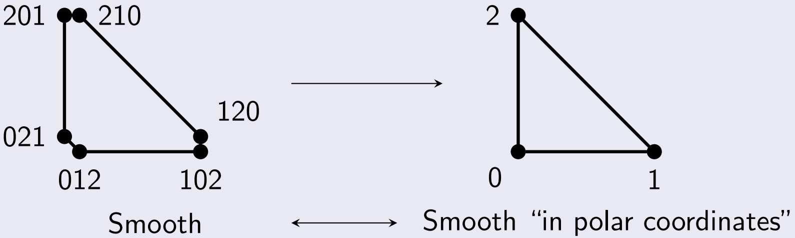
Blowing up manifolds with corners (Melrose, 1996)

- formalizes continuity/smoothness "in polar coordinates"



Section 4

Concluding remarks: Differential geometry vs. Riemannian geometry

Differential geometry vs. Riemannian geometry

Metric-independent finite element spaces

Differential geometry vs. Riemannian geometry

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- FEEC differential forms and their continuity conditions are defined without reference to a Riemannian metric.

Differential geometry vs. Riemannian geometry

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Metric-dependent finite element spaces

- Defining finite element spaces of vector fields with full continuity requires a Riemannian metric (even via differential form proxies).
- Behavior depends on whether angle defect is zero or not.

Thank you

Yakov Berchenko-Kogan and Evan S. Gawlik
Finite element spaces of double forms.
https://arxiv.org/abs/2505.17243

Yakov Berchenko-Kogan
Duality in finite element exterior calculus and Hodge duality on the sphere.
Found. Comput. Math. 21(5):1153–1180, 2021.

Evan S. Gawlik and Anil N. Hirani
Sequences from sequences, sans coordinates.
In preparation.

Yakov Berchenko-Kogan and Evan S. Gawlik
Blow-up Whitney forms, shadow forms, and Poisson processes.
Results in Applied Mathematics, special issue on Hilbert complexes, Paper No. 100529, 2025.

J. P. Brasselet, M. Goresky, and R. MacPherson.
Simplicial differential forms with poles.
Amer. J. Math., 113(6):1019–1052, 1991.

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