

Finite Element Spaces for Double Forms

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Tangential and normal continuity of vector fields

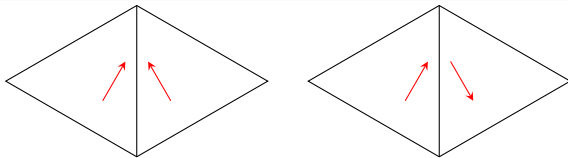


Figure: Tangential continuity (left) vs. normal continuity (right)

Tangential continuity

- Well-defined line integrals.
- In $H(\text{curl})$.

Normal continuity

- Well-defined fluxes.
- In $H(\text{div})$.

Full continuity

- Can yield spurious eigenvalues (AFW, 2010).

Matrix fields and tensor fields

Continuity conditions for matrix fields

- tangential–tangential
- normal–normal
- normal–tangential

Applications

- Strain/stress tensors
 - Elasticity (objects deforming under stress)
 - Fluid mechanics (Stokes equations)
- Curvature tensor
 - Numerical geometry
 - Numerical relativity

Differential forms corresponding to vector field $\langle M, N, P \rangle$

One-forms Λ^1

- $M dx + N dy + P dz$
- Restricted to the xy -plane $z = 0$:
 - $M dx + N dy$.
 - Tangential components.

Two-forms Λ^2

- $M dy \wedge dz + N dz \wedge dx + P dx \wedge dy$.
- Restricted to the xy -plane $z = 0$:
 - $P dx \wedge dy$.
 - Normal component.

Continuity conditions

- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are $(n - 1)$ -forms.

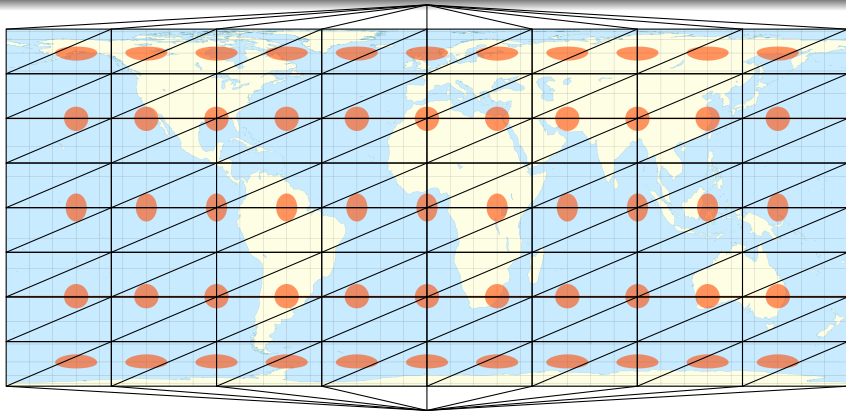
Vector fields (\mathbb{R}^3)

- Vector fields with tangential continuity are one-forms Λ^1 .
- Vector fields with normal continuity are two-forms Λ^2 .

Matrix fields ($\mathbb{R}^3 \otimes \mathbb{R}^3$)

- Matrix fields with tangential–tangential continuity are $(1, 1)$ -forms $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$.
- Matrix fields with normal–tangential continuity are $(2, 1)$ -forms $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$.
- Matrix fields with normal–normal continuity are $(2, 2)$ -forms $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$.

Intrinsic geometry with Regge metrics



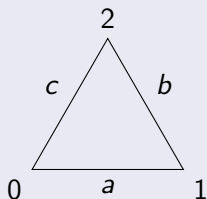
Regge finite elements

- Record the length of each edge.
- For each triangle, use the corresponding Euclidean metric.
- Get piecewise constant metric with tang.-tang. continuity.

Map credit: Wikipedia, Gaba

Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



Regge metric:

$$\begin{aligned} g = & -\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0) \\ & -\frac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) \\ & -\frac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2) \end{aligned}$$

Observations

- If \mathbf{v} is the vector from vertex 0 to vertex 1, then

$$d\lambda_0(\mathbf{v}) = -1, \quad d\lambda_1(\mathbf{v}) = 1, \quad d\lambda_2(\mathbf{v}) = 0.$$

As desired:

$$g(\mathbf{v}, \mathbf{v}) = -\frac{1}{2}a^2(-1 - 1) - \frac{1}{2}b^2(0 + 0) - \frac{1}{2}c^2(0 + 0) = a^2.$$

- Crucial: $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ is zero on other edges.

Constant coefficient finite elements for bilinear forms

Local bases for finite element spaces

- Each basis element φ must be associated to a face F of the triangulation, such that, for any other face G ,

$$\varphi \text{ is nonzero on } G \Leftrightarrow G \geq F.$$

Constant coefficient symmetric bilinear forms $\Lambda_{\text{sym}}^{1,1}$

- Regge's construction works in any dimension. To each edge ij , associate

$$d\lambda_i \otimes d\lambda_j + d\lambda_j \otimes d\lambda_i.$$

Constant coefficient antisymmetric bilinear forms $\Lambda_{\text{asym}}^{1,1}$

- Finite element spaces **do not exist** in dimension ≥ 3 .
- In 3D, antisymmetric bilinear forms \leftrightarrow vector fields with normal continuity.
- A nonzero constant vector field can't be tangent to three faces of a tetrahedron.

Natural subspaces of double forms

Theorem (Eigendecomposition of s^*s)

$$\Lambda^{p,q} = \bigoplus_m \Lambda_m^{p,q}, \quad \max\{0, q - p\} \leq m \leq \min\{q, n - p\}.$$

Example

- $\Lambda_0^{1,1}$: Symmetric bilinear forms, $\varphi(X; Y) = \varphi(Y; X)$.
- $\Lambda_1^{1,1}$: Λ^2 , antisymmetric bilinear forms, $\varphi(X; Y) = -\varphi(Y; X)$.

- $\Lambda_0^{2,1}$: spanned by $\alpha \otimes \beta$ such that $\alpha \wedge \beta = 0$.
 - Matrix proxy in 3D: trace-free matrices.
- $\Lambda_1^{2,1}$: Λ^3 .
 - Matrix proxy in 3D: multiples of the identity matrix.

- $\Lambda_0^{2,2}$: Symmetric, satisfying the algebraic Bianchi identity.
 - Riemann curvature tensor.
- $\Lambda_1^{2,2}$: Antisymmetric, $\varphi(X, Y; Z, W) = -\varphi(Z, W; X, Y)$.
- $\Lambda_2^{2,2}$: Λ^4 .

Theorem

Apart from $\Lambda_q^{p,q} \cong \Lambda^{p+q}$ with constant coefficients, there is a finite element space for every natural space of double forms $\Lambda_m^{p,q}$ with polynomial coefficients of any degree (including zero).

Example (Constant coefficient spaces)

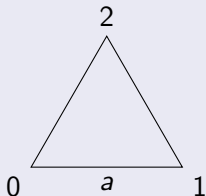
- $\Lambda_0^{1,1}$: symmetric matrices with tangential–tangential continuity (Regge, 1961).
 - Higher order: (Li, 2018).
- $\Lambda_0^{2,1}$ in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda_0^{2,2}$ in 3D: symmetric matrices with normal–normal continuity (Pechstein and Schöberl, 2011).
- $\Lambda_0^{2,2}$ (or $\Lambda_0^{n-2,n-2}$) in any dimension: finite elements for the Riemann curvature tensor.

Degrees of freedom for constant coefficient spaces

	d						
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	1	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	2	0	0	0	0
$\Lambda_0^{2,2}$	0	0	1	2	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	3	0	0	0
$\Lambda_0^{3,2}$	0	0	0	3	5	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	4	0	0
$\Lambda_0^{3,3}$	0	0	0	1	5	5	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	6	9	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	5	0

Table: Number of degrees of freedom for $\Lambda^{p,q}$ associated to a face of the triangulation of dimension d is $\frac{p-q+2m+1}{p+m+1} \binom{d+1}{q-m} \binom{q-m-1}{d-p-m}$.

Recall



- It was crucial that $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ vanishes on the other edges.

Extension operators

- We need to be able to take a form on edge 01, and extend it to the triangle so that it vanishes on the other edges.
- The metric on edge 01 is $a^2 d\lambda_1 \otimes d\lambda_1$.
- However, if we extend to the triangle using the formula $a^2 d\lambda_1 \otimes d\lambda_1$, it won't vanish on edge 12.
- We first need to use $d\lambda_0 + d\lambda_1 = 0$ to rewrite $a^2 d\lambda_1 \otimes d\lambda_1$ as $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ on edge 01.

Constructing extensions

Example $(\mathcal{P}_r \Lambda_m^{p,q} = \mathcal{P}_0 \Lambda_0^{1,1})$

- ❶ Start with a form on edge 01 with vanishing trace: $d\lambda_1 \otimes d\lambda_1$
- ❷ $\lambda_i = u_i^2, \quad d\lambda_i = 2u_i du_i: \quad 4u_1^2 du_1 \otimes du_1.$
- ❸ $u_0 du_0 + u_1 du_1$ wedge with each factor:
 $4u_0^2 u_1^2 (du_0 \wedge du_1) \otimes (du_0 \wedge du_1).$
- ❹ Hodge star both factors (as forms on \mathbb{R}^2): $4u_0^2 u_1^2.$
- ❺ Divide by $u_0 u_1$: $4u_0 u_1.$
- ❻ Divide by $(2r + p + m + 1)(2r + q - m) = 2$: $2u_0 u_1.$
- ❼ Exterior derivative on both factors: $2(du_0 \otimes du_1 + du_1 \otimes du_0).$
- ❽ Apply $(-1)^{p+q}$ times the inverse Hodge star:
 $-2(du_1 \otimes du_0 + du_0 \otimes du_1).$
- ❾ Multiply by $u_0 u_1$: $-2u_0 u_1 (du_1 \otimes du_0 + du_0 \otimes du_1).$
- ❿ Convert back to λ_i : $-\frac{1}{2}(d\lambda_1 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_1).$

Thank you



D. N. Arnold, R. S. Falk, and R. Winther

Finite element exterior calculus: from Hodge theory to numerical stability.

Bull. Amer. Math. Soc. (N.S.), 47(2):281–354, 2010.



J. Gopalakrishnan, P. L. Lederer, and J. Schöberl

A mass conserving mixed stress formulation for the Stokes equations.

IMA J. Num. Analysis, 40(3):1838–1874, 2020.



L. Li

Regge finite elements with applications in solid mechanics and relativity.

PhD thesis. University of Minnesota, 2018.



A. Pechstein and J. Schöberl.

Tangential-displacement and normal–normal-stress continuous mixed finite elements for elasticity.

Math. Models and Methods in Appl. Sci. 21.08:1761–1782, 2011.

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