Finite Element Spaces for Double Forms

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Tangential and normal continuity of vector fields

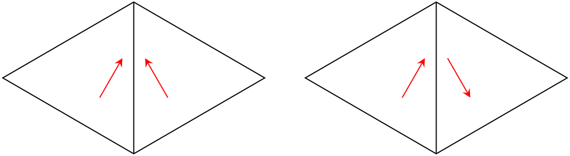


Figure: Tangential continuity (left) vs. normal continuity (right)

Tangential continuity

- Well-defined line integrals.
- In $H(\mathrm{curl})$.

Normal continuity

- Well-defined fluxes.
- In $H(\mathrm{div})$.

Full continuity

- Can yield spurious eigenvalues (AFW, 2010).

Matrix fields and tensor fields

Continuity conditions for matrix fields

- tangential–tangential
- normal–normal
- normal–tangential

Applications

- Strain/stress tensors
  - Elasticity (objects deforming under stress)
  - Fluid mechanics (Stokes equations)
- Curvature tensor
  - Numerical geometry
  - Numerical relativity

Differential forms corresponding to vector field $\langle M, N, P\rangle$

One-forms Λ^1

- $M\,dx + N\,dy + P\,dz$
- Restricted to the xy-plane $z = 0$:
  - $M\,dx + N\,dy$.
  - Tangential components.

Two-forms Λ^2

- $M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$.
- Restricted to the xy-plane $z = 0$:
  - $P\,dx \wedge dy$.
  - Normal component.

Continuity conditions

- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are $(n-1)$-forms.

Double forms

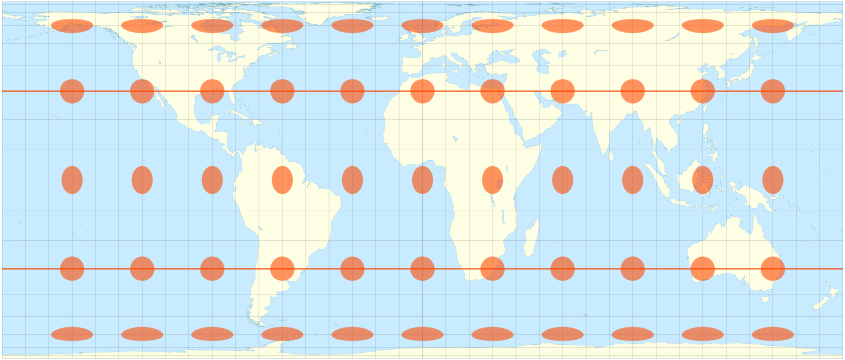
Vector fields ($\mathbb{R}^3$)

- Vector fields with tangential continuity are one-forms Λ^1.
- Vector fields with normal continuity are two-forms Λ^2.

Matrix fields ($\mathbb{R}^3 \otimes \mathbb{R}^3$)

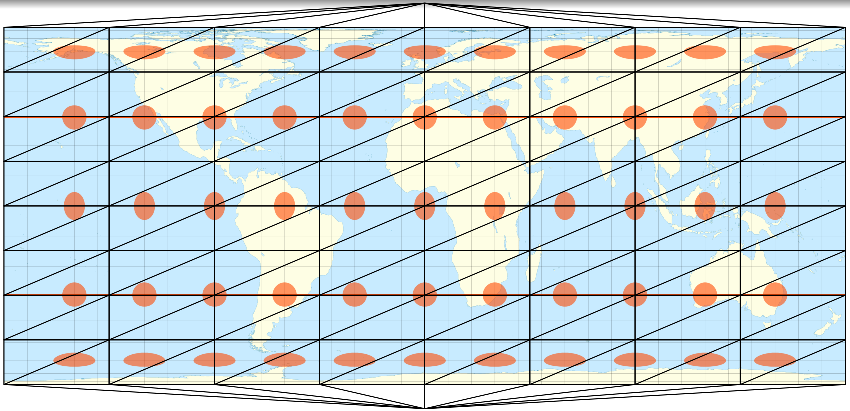
- Matrix fields with tangential–tangential continuity are $(1,1)$-forms $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$.
- Matrix fields with normal–tangential continuity are $(2,1)$-forms $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$.
- Matrix fields with normal–normal continuity are $(2,2)$-forms $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$.

Intrinsic geometry with Regge metrics



Map credit: Wikipedia, Gaba

Intrinsic geometry with Regge metrics



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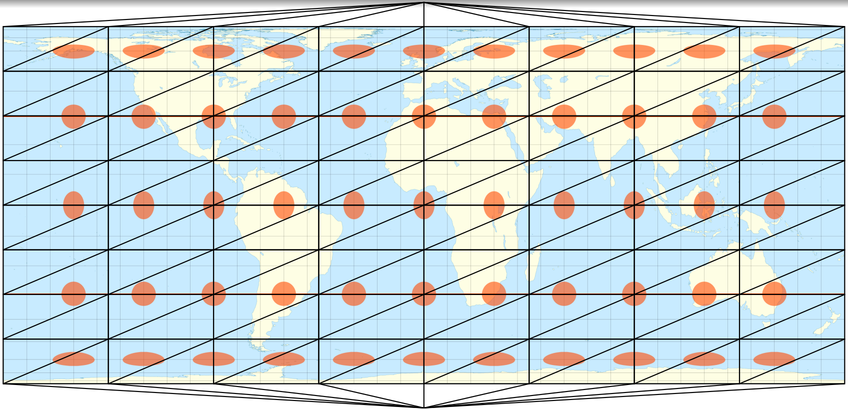
Intrinsic geometry with Regge metrics



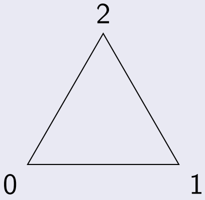
Regge finite elements

- Record the length of each edge.
- For each triangle, use the corresponding Euclidean metric.
- Get piecewise constant metric with tang.–tang. continuity.

Map credit: Wikipedia, Gaba

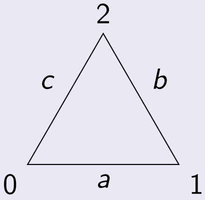
Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



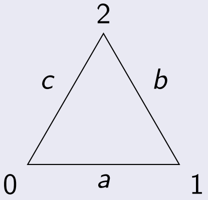
Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$

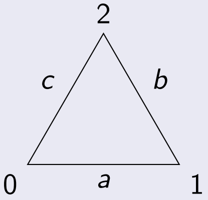


Regge metric:

$$
\begin{aligned}
g = &-\tfrac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)\\
&-\tfrac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1)\\
&-\tfrac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2)
\end{aligned}
$$

Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



Regge metric:

$$g = -\tfrac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0) - \tfrac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) - \tfrac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2)$$

Observations

- If $\mathbf{v}$ is the vector from vertex 0 to vertex 1, then

$$d\lambda_0(\mathbf{v}) = -1, \qquad d\lambda_1(\mathbf{v}) = 1, \qquad d\lambda_2(\mathbf{v}) = 0.$$

  As desired:

$$g(\mathbf{v},\mathbf{v}) = -\tfrac{1}{2}a^2(-1-1) - \tfrac{1}{2}b^2(0+0) - \tfrac{1}{2}c^2(0+0) = a^2.$$

- Crucial: $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ is zero on other edges.

Constant coefficient finite elements for bilinear forms

Local bases for finite element spaces

- Each basis element φ must be associated to a face F of the triangulation, such that, for any other face G,

$$\varphi \text{ is nonzero on } G \Leftrightarrow G \geq F.$$

Constant coefficient symmetric bilinear forms $\Lambda^{1,1}_{\mathrm{sym}}$

- Regge's construction works in any dimension. To each edge ij, associate

$$d\lambda_i \otimes d\lambda_j + d\lambda_j \otimes d\lambda_i.$$

Constant coefficient antisymmetric bilinear forms $\Lambda^{1,1}_{\mathrm{asym}}$

- Finite element spaces do not exist in dimension ≥ 3.
- In 3D, antisymmetric bilinear forms $\leftrightarrow$ vector fields with normal continuity.
- A nonzero constant vector field can't be tangent to three faces of a tetrahedron.

Natural subspaces of double forms

Theorem (Eigendecomposition of s^*s)

$$\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m, \qquad \max\{0, q-p\} \le m \le \min\{q, n-p\}.$$

Example

- $\Lambda^{1,1}_0$: Symmetric bilinear forms, $\varphi(X;Y) = \varphi(Y;X)$.
- $\Lambda^{1,1}_1$: Λ^2, antisymmetric bilinear forms, $\varphi(X;Y) = -\varphi(Y;X)$.

- $\Lambda^{2,1}_0$: spanned by $\alpha \otimes \beta$ such that $\alpha \wedge \beta = 0$.
  - Matrix proxy in 3D: trace-free matrices.
- $\Lambda^{2,1}_1$: Λ^3.
  - Matrix proxy in 3D: multiples of the identity matrix.

- $\Lambda^{2,2}_0$: Symmetric, satisfying the algebraic Bianchi identity.
  - Riemann curvature tensor.
- $\Lambda^{2,2}_1$: Antisymmetric, $\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$.
- $\Lambda^{2,2}_2$: Λ^4.

Finite element spaces

Theorem

Apart from $\Lambda_q^{p,q} \cong \Lambda^{p+q}$ with constant coefficients, there is a finite element space for every natural space of double forms $\Lambda_m^{p,q}$ with polynomial coefficients of any degree (including zero).

Example (Constant coefficient spaces)

- $\Lambda_0^{1,1}$: symmetric matrices with tangential–tangential continuity (Regge, 1961).
  - Higher order: (Li, 2018).
- $\Lambda_0^{2,1}$ in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda_0^{2,2}$ in 3D: symmetric matrices with normal–normal continuity (Pechstein and Schöberl, 2011).
- $\Lambda_0^{2,2}$ (or $\Lambda_0^{n-2,n-2}$) in any dimension: finite elements for the Riemann curvature tensor.

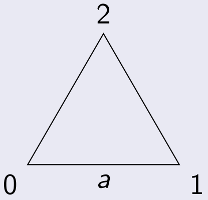
Degrees of freedom for constant coefficient spaces

	d						
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	**1**	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	**2**	0	0	0	0
$\Lambda_0^{2,2}$	0	0	**1**	**2**	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	**3**	0	0	0
$\Lambda_0^{3,2}$	0	0	0	**3**	**5**	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	**4**	0	0
$\Lambda_0^{3,3}$	0	0	0	**1**	**5**	**5**	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	**6**	**9**	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	**5**	0

Table: Number of degrees of freedom for $\Lambda_m^{p,q}$ associated to a face of the triangulation of dimension d is $\frac{p-q+2m+1}{p+m+1}\binom{d+1}{q-m}\binom{q-m-1}{d-p-m}$.

Extension

Recall



- It was crucial that $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ vanishes on the other edges.

Extension operators

- We need to be able to take a form on edge 01, and extend it to the triangle so that it vanishes on the other edges.
- The metric on edge 01 is $a^2\, d\lambda_1 \otimes d\lambda_1$.
- However, if we extend to the triangle using the formula $a^2\, d\lambda_1 \otimes d\lambda_1$, it won't vanish on edge 12.
- We first need to use $d\lambda_0 + d\lambda_1 = 0$ to rewrite $a^2\, d\lambda_1 \otimes d\lambda_1$ as $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ on edge 01.

Constructing extensions

Example ($\mathcal{P}_r\Lambda^{p,q}_m = \mathcal{P}_0\Lambda^{1,1}_0$)

1. Start with a form on edge 01 with vanishing trace: $d\lambda_1 \otimes d\lambda_1$
2. $\lambda_i = u_i^2, \quad d\lambda_i = 2u_i\,du_i$: $4u_1^2\,du_1 \otimes du_1$.
3. $u_0\,du_0 + u_1\,du_1$ wedge with each factor: $4u_0^2u_1^2(du_0 \wedge du_1) \otimes (du_0 \wedge du_1)$.
4. Hodge star both factors (as forms on $\mathbb{R}^2$): $4u_0^2u_1^2$.
5. Divide by u_0u_1: $4u_0u_1$.
6. Divide by $(2r+p+m+1)(2r+q-m) = 2$: $2u_0u_1$.
7. Exterior derivative on both factors: $2(du_0 \otimes du_1 + du_1 \otimes du_0)$.
8. Apply $(-1)^{p+q}$ times the inverse Hodge star: $-2(du_1 \otimes du_0 + du_0 \otimes du_1)$.
9. Multiply by u_0u_1: $-2u_0u_1(du_1 \otimes du_0 + du_0 \otimes du_1)$.
10. Convert back to λ_i: $-\frac{1}{2}(d\lambda_1 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_1)$.

Thank you

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