Numerically Computing the Entropy and Index of Mean Curvature Flow Self-Shrinkers

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Penn State

Outline

Section 1

Introduction to Mean Curvature Flow Self-Shrinkers

Curve shortening flow

$$\frac{d}{dt}\mathbf{x} = -\kappa(\mathbf{x})\mathbf{n}.$$

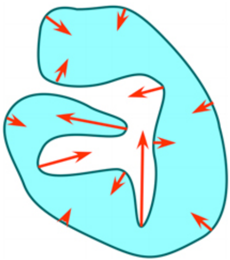

Figure: Curve shortening flow. Image credit: Treibergs, 2010 slides. Video credit: Angenent, 2011 YouTube.

Mean curvature flow

$$\frac{d}{dt}\mathbf{x} = -H(\mathbf{x})\mathbf{n}$$

Figure: Mean curvature flow. Video credit: Kovács, Li, Lubich, 2019.

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  - Yes, a torus (Angenent, 1989).

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  - Many others (see papers by Drugan, Kapouleas, Kleene, Lee, McGrath, Møller, Nguyen, etc.).

The Angenent torus

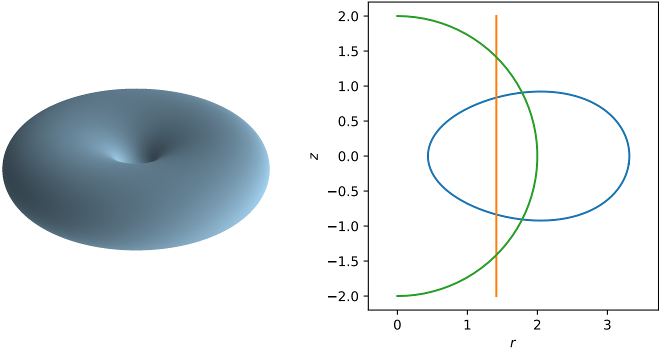

Figure: The Angenent torus (left) and its cross-section (right), with the self-shrinking sphere (green) and cylinder (orange) for comparison.

Angenent torus intuition

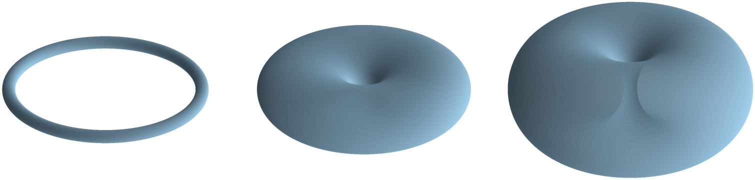

Figure: Meridian collapse (left), inner longitude collapse (right), just right (middle).

Section 2

Entropy

A variational formulation for self-shrinkers

Theorem (Huisken, 1990)

A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is a self-shrinker that becomes extinct at the origin after one unit of time if and only if it is a critical point of the weighted area functional called the F-functional.

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i.e. for any family of surfaces Σ_s parametrized by s with $\Sigma_0 = \Sigma$, we have

$$\left.\frac{d}{ds}\right|_{s=0} F(\Sigma_s) = 0.$$

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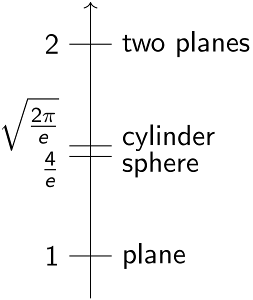
The critical value of the F-functional, called the entropy of the self-shrinker, is helpful in understanding what kinds of singularities can occur.



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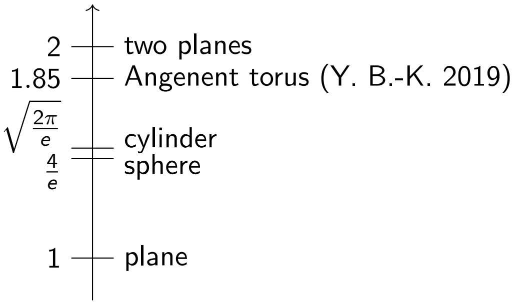

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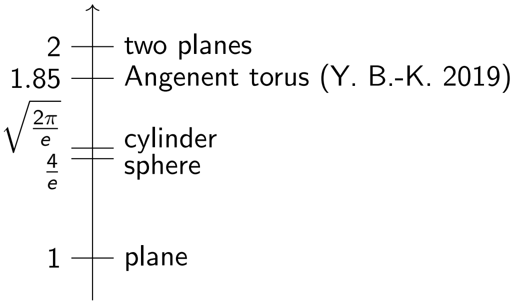

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Earlier work (Drugan and Nguyen, 2018): the entropy of the Angenent torus is less than 2.

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- Continuing mean curvature flow past a singularity; see work of Mramor and Wang, 2018.

Numerical estimates of the entropy of the Angenent torus

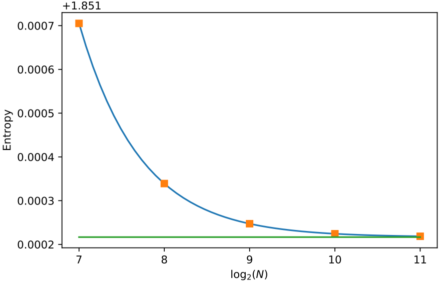

Figure: The entropy of the Angenent torus as computed using 128, 256, 512, 1024, and 2048 points. The values (orange) appear to lie on an exponential curve (blue) converging to 1.8512167 (green).

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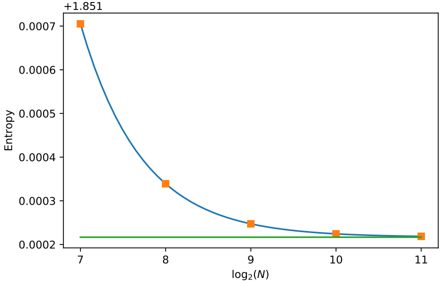

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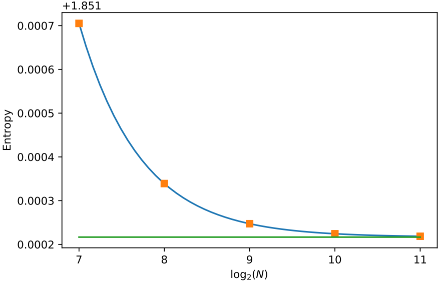

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- Later work (Barrett, Deckelnick, Nürnberg, 2020) obtained the same value using different methods.

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- The number of independent perturbations of a critical point that decrease the value of a functional is called the (Morse) index of the critical point.

Toy example illustrating stability

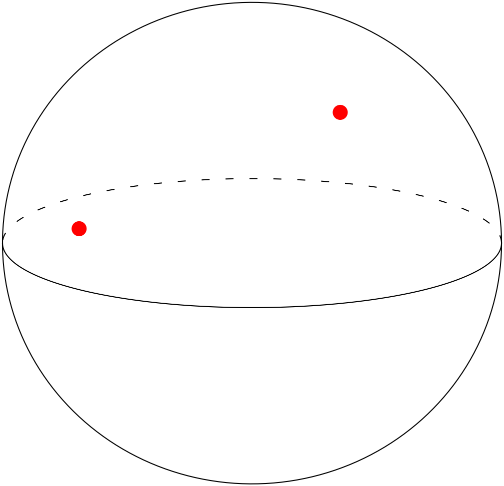

Figure: Two cities can be connected with a stable geodesic and with an unstable geodesic.

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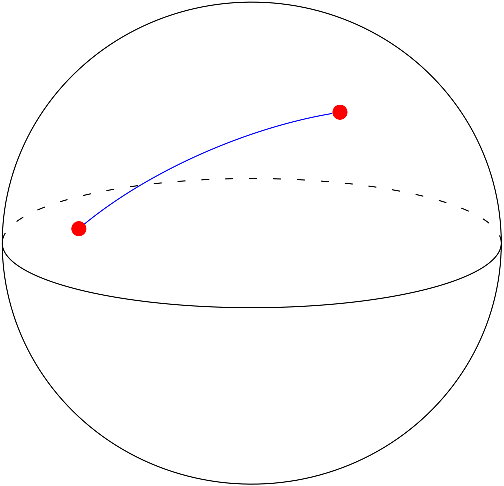

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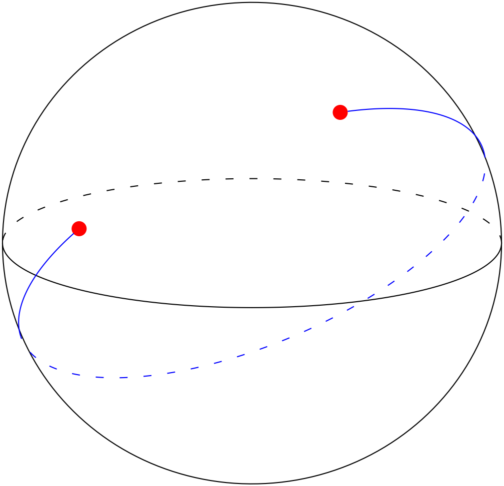

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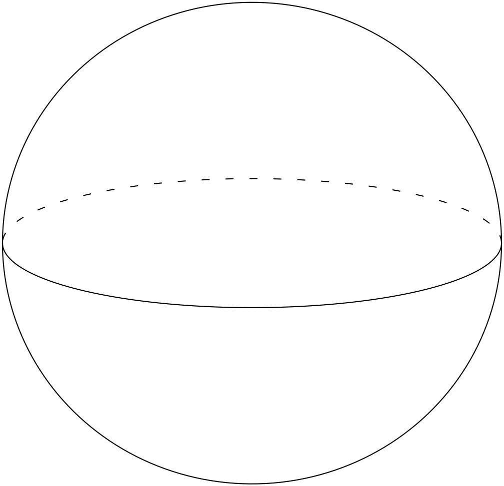

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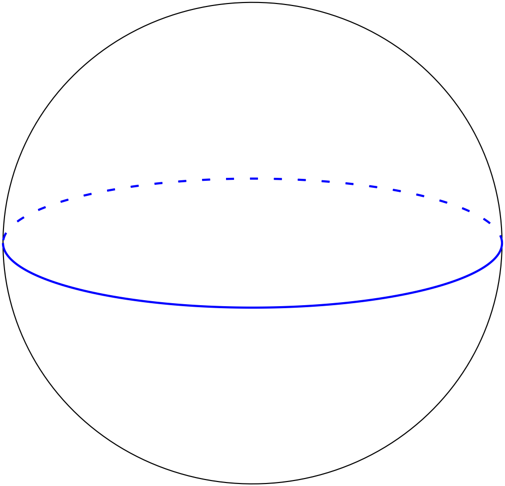

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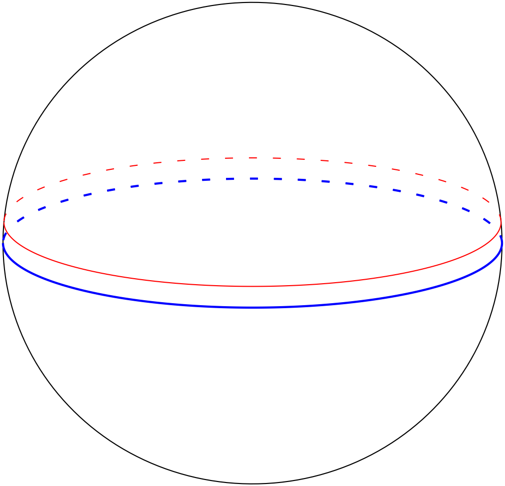

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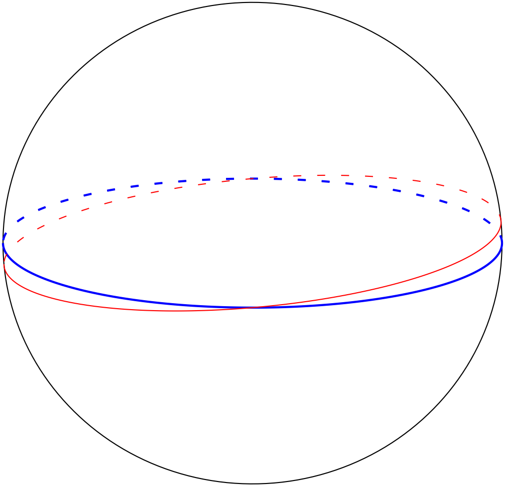

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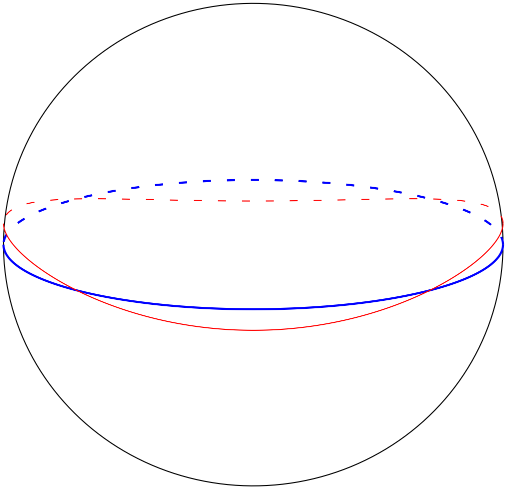

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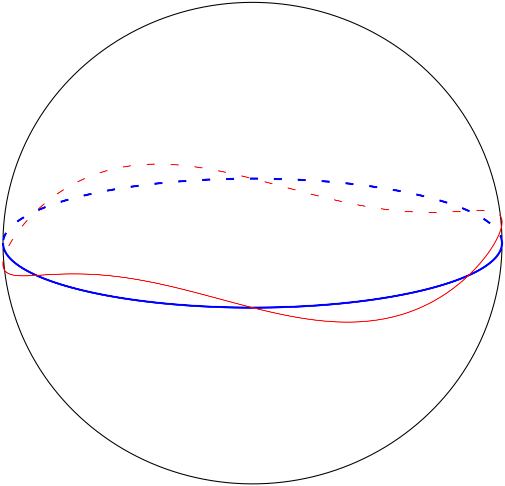

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- At least three other variations exist (Liu, 2016).

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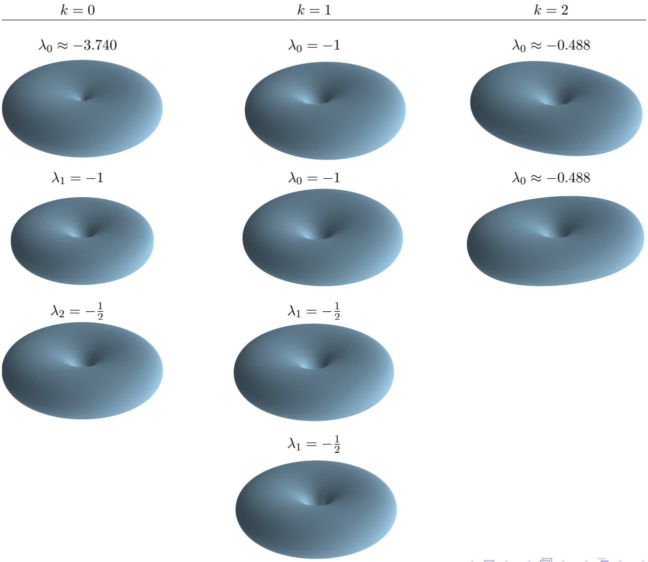
- This lets us compute variations of the cross-section (1D) rather than variations of the surface (2D).

Index results (Y. B.-K. 2020)



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- Other work giving lower bounds: see (McGonagle, 2015; Liu, 2016; Aiex, 2019).

Section 4

References and Future Work

References

Yakov Berchenko-Kogan.
The entropy of the Angenent torus is approximately 1.85122.
Experimental Math., 2019.

Yakov Berchenko-Kogan.
Bounds on the index of rotationally symmetric self-shrinking tori.
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- Prove bounds on the accuracy of the entropy values and the eigenvalues.
  - Starting point: numerical analysis theorems about convergence rates of approximations to minimal surfaces. See Dziuk and Hutchinson, 1999; Dierkes, Jenschke, Pozzi, 2018.
  - We have a weight (the F-functional) and we need not just convergence rates but actual bounds, but the same techniques apply (fundamentally, Taylor's theorem).

Thank you

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