The Combinatorics of Finite Element Methods

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Outline

1. From finite elements to the Euler characteristic.

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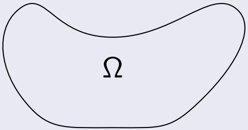
   - Finite element spaces that do not respect cohomology might give wrong answers.

Numerically solving PDEs

Sample Problem



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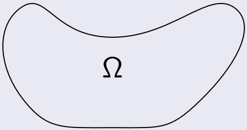

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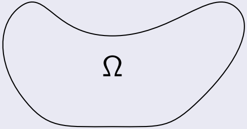

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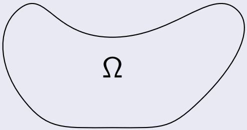

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- To solve numerically, we must discretize.

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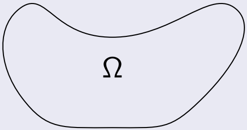
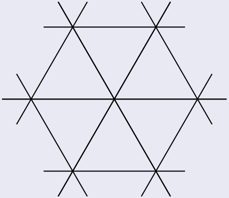


Discretization

- To solve numerically, we must discretize.
- We need a finite-dimensional space of functions that "approximates" the full infinite-dimensional space of possible u.

Finite-dimensional function spaces

Continuous piecewise linear functions to $\mathbb{R}$



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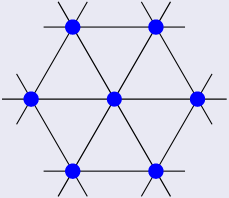

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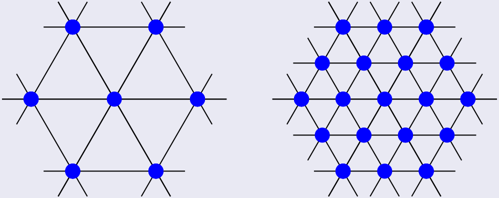

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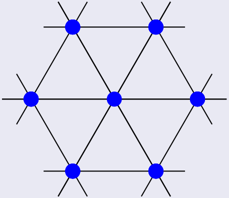

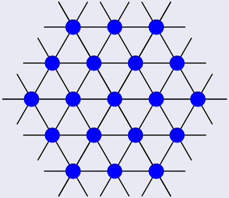

Continuous piecewise polynomial functions to $\mathbb{R}$

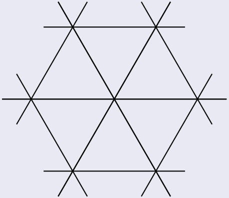

Figure: Piecewise quadratic (left) and piecewise cubic (right)

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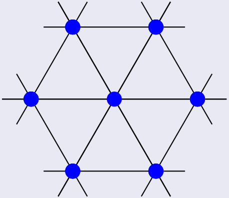

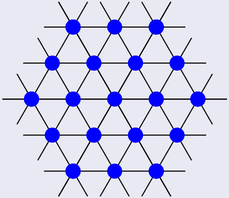

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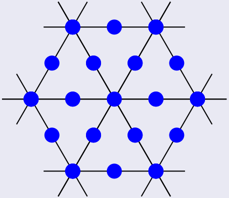

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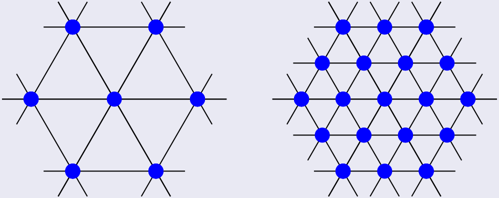

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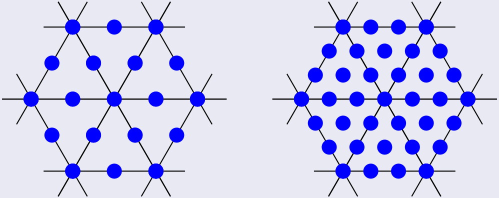

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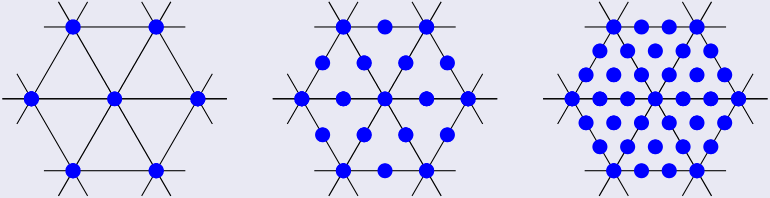
Degrees of freedom

Piecewise linear/quadratic/cubic continuous scalar-valued functions



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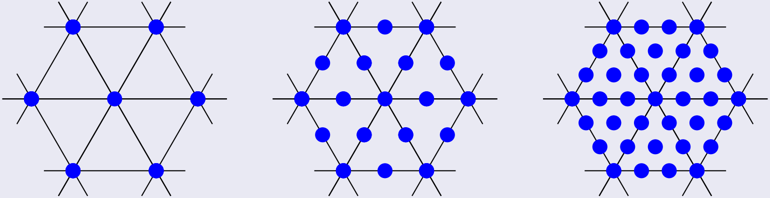

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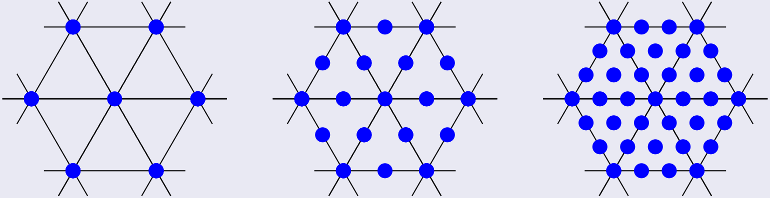

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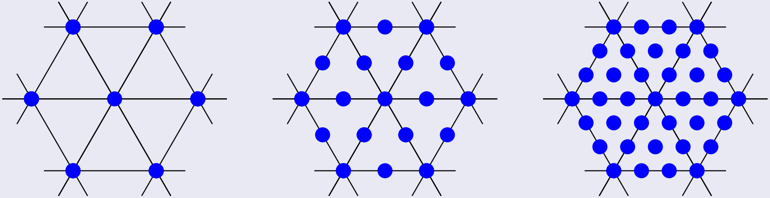

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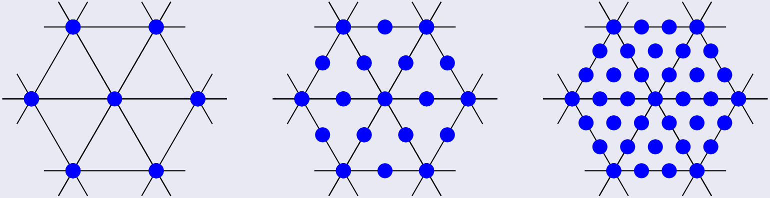

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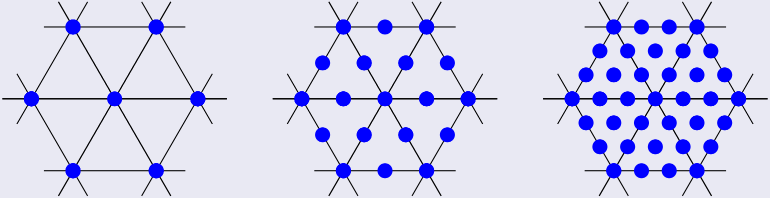

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Piecewise linear	$\mathbb{R}^{V}$
Piecewise quadratic	$\mathbb{R}^{V+E}$
Piecewise cubic	$\mathbb{R}^{V+2E+F}$

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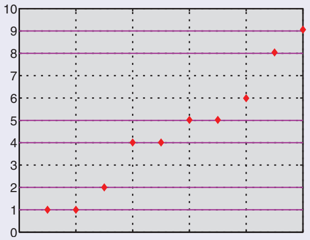

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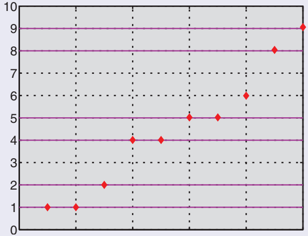

- Using vector fields with full continuity yields false eigenvalue $\lambda = 6$.
- To get the right eigenvalues, we need better finite element spaces of vector fields.

Gradients of piecewise smooth scalar fields

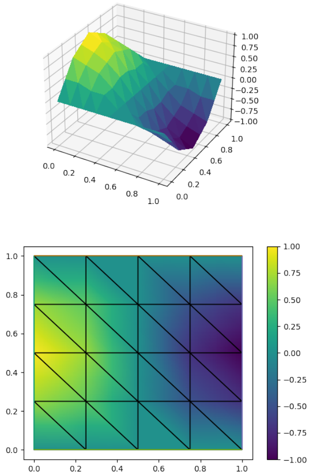

Figure: A piecewise linear function (left) and its gradient (right)

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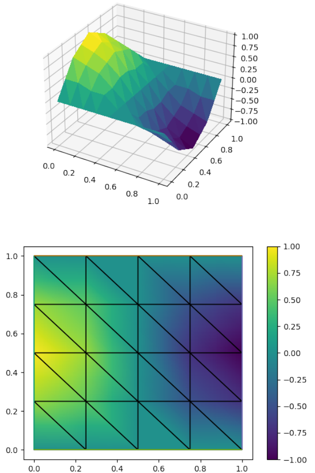

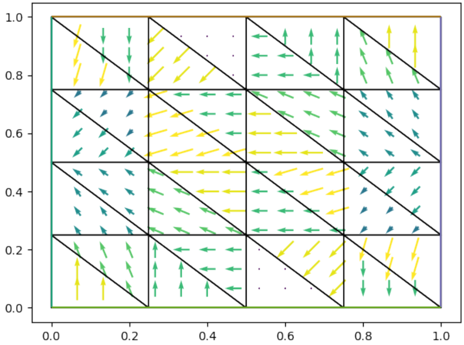

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Finite-dimensional spaces of vector fields

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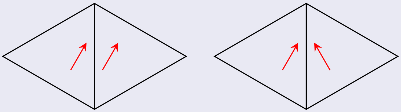

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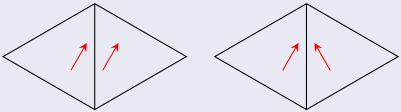

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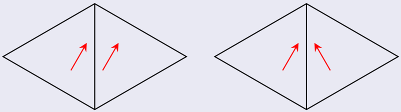

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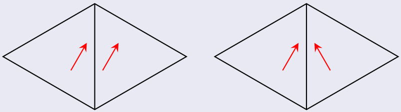

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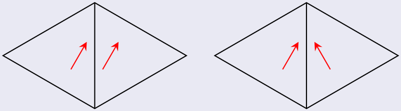

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  - Having well-defined line integrals requires only tangential continuity.

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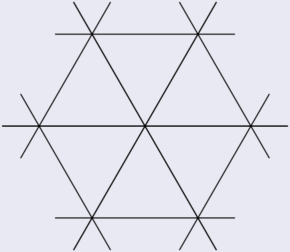

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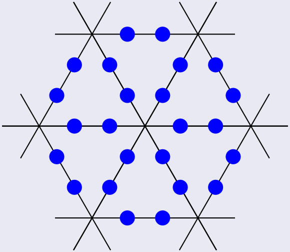

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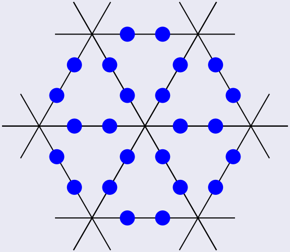

Higher degree?

Periodic Table of the Finite Elements

Complexes

A discrete complex

$$\begin{matrix}\text{continuous}\\\text{piecewise cubic}\\\text{scalar fields}\end{matrix} \xrightarrow{\text{grad}} \begin{matrix}\text{tangentially continuous}\\\text{piecewise quadratic}\\\text{vector fields}\end{matrix} \xrightarrow{\text{curl}} \begin{matrix}\text{discontinuous}\\\text{piecewise linear}\\\text{scalar fields}\end{matrix}$$

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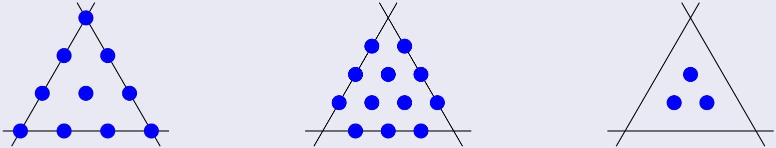
continuous piecewise cubic scalar fields $\xrightarrow{\text{grad}}$ tangentially continuous piecewise quadratic vector fields $\xrightarrow{\text{curl}}$ discontinuous piecewise linear scalar fields



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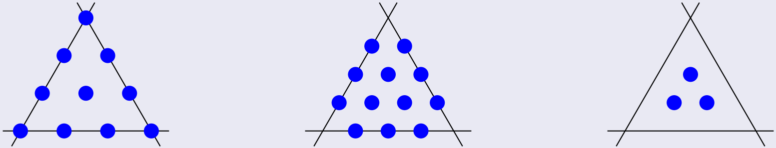

$$\mathbb{R}^{V+2E+F} \longrightarrow \mathbb{R}^{3E+3F} \longrightarrow \mathbb{R}^{3F}$$

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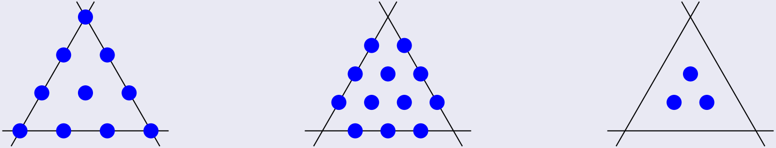

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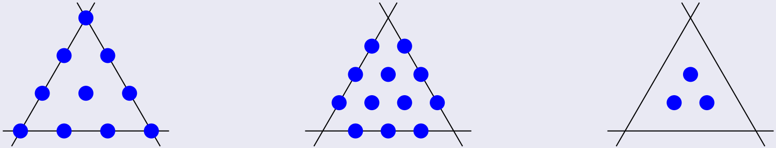

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Euler characteristic

- This complex has the right Euler characteristic:

$$(V + 2E + F) - (3E + 3F) + 3F = V - E + F.$$

Euler characteristic

$V - E + F = 2$ (Maurolico, 1537)

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Figure: Wikipedia, "Euler characteristic"

Works for all convex polyhedra

Soccer ball:

$$V - E + F = 60 - 90 + 32 = 2.$$

Euler characteristic for other shapes

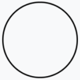
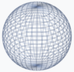
Name	Image	χ
Interval		**1**
Circle		**0**
Disk		**1**
Sphere		**2**
Torus *(Product of two circles)*		**0**
Double torus		**−2**
Triple torus		**−4**

Figure: Wikipedia, "Euler characteristic"

Vector calculus in the plane (or on a surface)

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

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$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

- If $E = \operatorname{grad} \phi$, then $\operatorname{curl} E = 0$.

Vector calculus in the plane (or on a surface)

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

- If $E = \operatorname{grad} \phi$, then $\operatorname{curl} E = 0$.
- If $\operatorname{curl} E = 0$, then $E = \operatorname{grad} \phi$ for some ϕ.

Vector calculus in the plane (or on a surface)

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

- If $E = \operatorname{grad}\phi$, then $\operatorname{curl} E = 0$. always true
- If $\operatorname{curl} E = 0$, then $E = \operatorname{grad}\phi$ for some ϕ. not always true

curl $E = 0$ but $E \neq \operatorname{grad} \phi$

The electric field around a solenoid

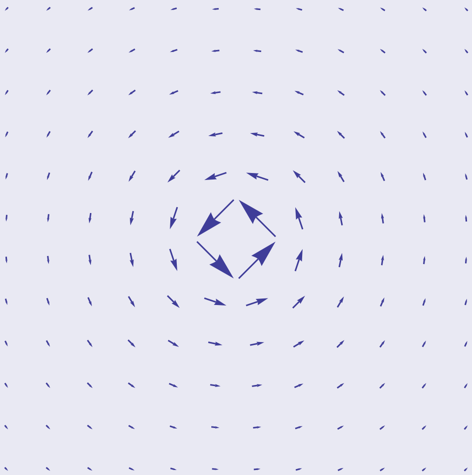

Figure: Wikipedia, "Irrotational vector field"

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

The first cohomology group H^1

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

The first cohomology group H^1

- Informally, the first cohomology group of a domain Ω is the set of counterexamples:

de Rham cohomology

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- Informally, the first cohomology group of a domain Ω is the set of counterexamples:
    - Vector fields E on Ω

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The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

The first cohomology group H^1

- Informally, the first cohomology group of a domain Ω is the set of counterexamples:
  - Vector fields E on Ω
    - whose curls are zero, but

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    - Vector fields E on Ω
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- Caveat: If E is a counterexample, then so is $E' := E + \operatorname{grad} \psi$.

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- In the first cohomology group H^1, we view E and E' as "equivalent counterexamples".

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  - $\operatorname{curl} E' = \operatorname{curl} E + 0 = 0$.
  - If E is not a gradient then neither is E'.
- In the first cohomology group H^1, we view E and E' as "equivalent counterexamples".
- $\dim H^1$ counts the number of "holes" in the domain.

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology

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de Rham cohomology, informally

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^1: vector fields E whose curls are zero but which aren't gradients.

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The zeroth cohomology group H^0

de Rham cohomology

The de Rham complex

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de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The zeroth cohomology group H^0

- If $\operatorname{grad} \phi = 0$ then ϕ is constant.

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The zeroth cohomology group H^0

- If $\operatorname{grad}\phi = 0$ then ϕ is constant.
- So $\dim H^0 = 1$.

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The zeroth cohomology group H^0

- If $\operatorname{grad} \phi = 0$ then ϕ is constant only for connected domains.
- So $\dim H^0 = 1$ for connected domains.

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The zeroth cohomology group H^0

- If $\operatorname{grad}\phi = 0$ then ϕ is constant only for connected domains.
- So $\dim H^0 = 1$ for connected domains.
- $\dim H^0$ counts the number of connected components of the domain.

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The second cohomology group H^2

de Rham cohomology

The de Rham complex

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de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The second cohomology group H^2

- For planar domains $H^2 = 0$ (every scalar field is a curl).

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The second cohomology group H^2

- For planar domains $H^2 = 0$ (every scalar field is a curl).
- For a closed surface S (e.g. sphere), H^2 is the constants.

de Rham cohomology

The de Rham complex

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

de Rham cohomology, informally

- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
- H^2: scalar fields ρ which aren't curls.

The second cohomology group H^2

- For planar domains $H^2 = 0$ (every scalar field is a curl).
- For a closed surface S (e.g. sphere), H^2 is the constants.
  - If B is tangent to S then $\int_S \operatorname{curl} B = 0$ by Stokes's theorem.

de Rham cohomology

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- H^0: scalar fields ϕ whose gradients are zero.
- H^1: vector fields E whose curls are zero but which aren't gradients.
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The second cohomology group H^2

- For planar domains $H^2 = 0$ (every scalar field is a curl).
- For a closed surface S (e.g. sphere), H^2 is the constants.
  - If B is tangent to S then $\int_S \operatorname{curl} B = 0$ by Stokes's theorem.
  - But $\int_S 1 \neq 0$, so 1 is not a curl.

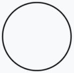
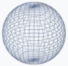
From Euler characteristic to cohomology (1930s)

The continuous setting

Cohomology tells you the Euler characteristic

The Euler characteristic is

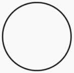
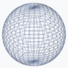
$$V - E + F,$$

Name	Image	χ
Interval		1
Circle		0
Disk		1
Sphere		2
Torus *(Product of two circles)*		0
Double torus		−2
Triple torus		−4

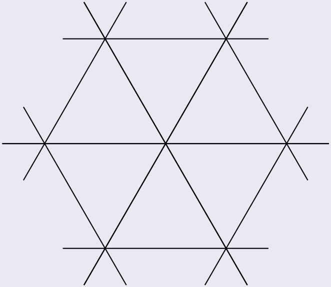
Cohomology tells you the Euler characteristic

The Euler characteristic is

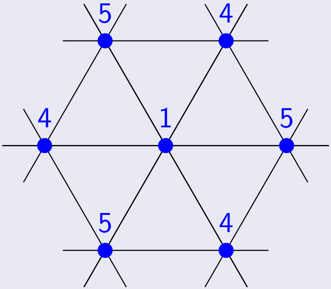
$$\begin{array}{ccccc} V & - & E & + & F, \\ \dim H^0 & - & \dim H^1 & + & \dim H^2. \end{array}$$

Name	Image	χ
Interval		1
Circle		0
Disk		1
Sphere		2
Torus *(Product of two circles)*		0
Double torus		−2
Triple torus		−4

Discrete gradient



Discrete gradient



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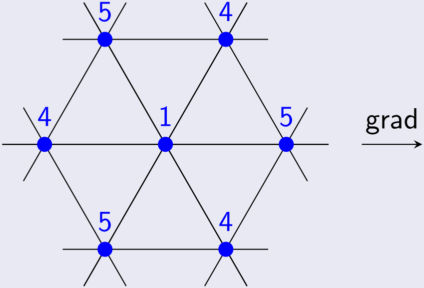
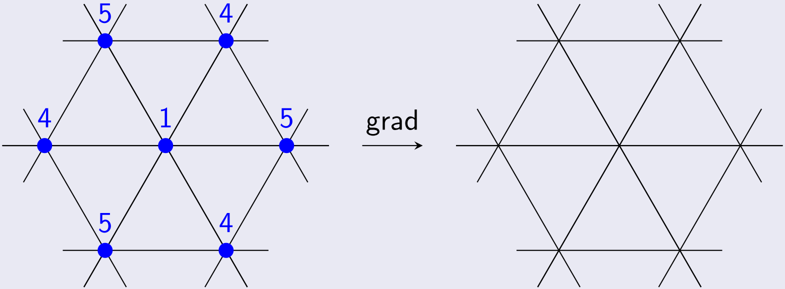


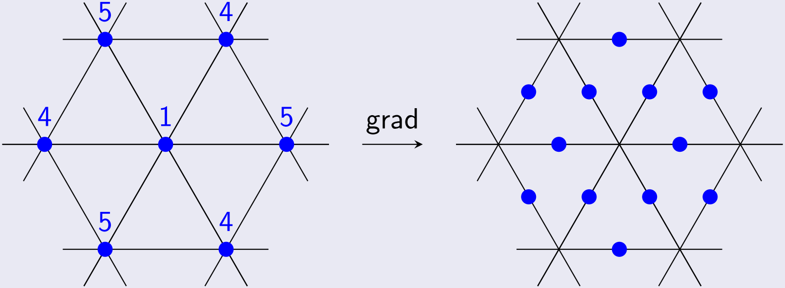
Discrete gradient



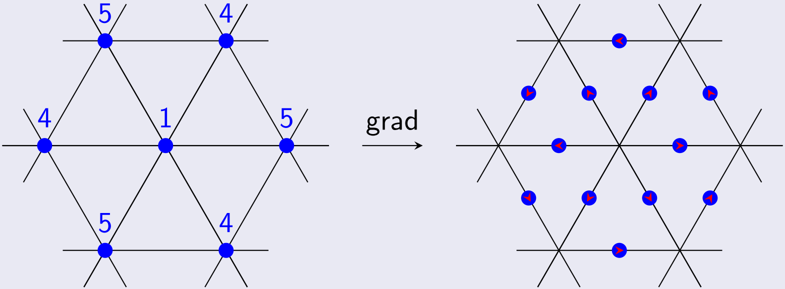
From Euler characteristic to cohomology (1930s)

The discrete setting

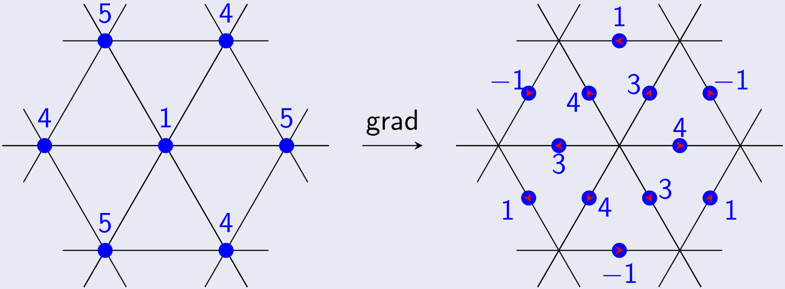
Discrete gradient



Discrete gradient



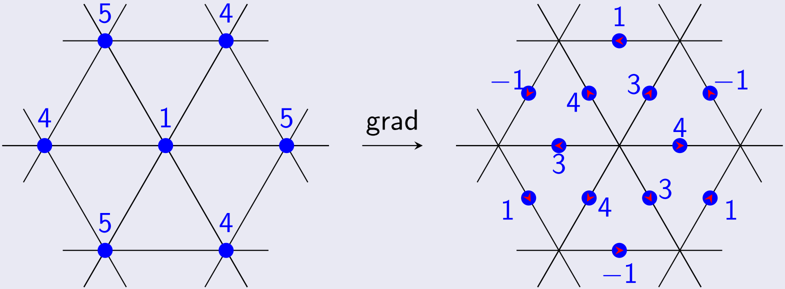
Discrete gradient



From Euler characteristic to cohomology (1930s)

The discrete setting

Discrete gradient



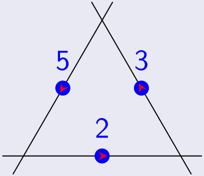
Fundamental theorem of line integrals

$$\int_C \operatorname{grad} \phi = \phi \Big|_{v_0}^{v_1}$$

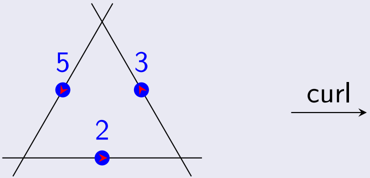
for a curve C from point v_0 to point v_1.

Discrete curl

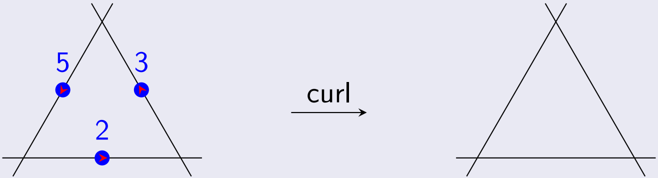
Discrete curl



Discrete curl



Discrete curl



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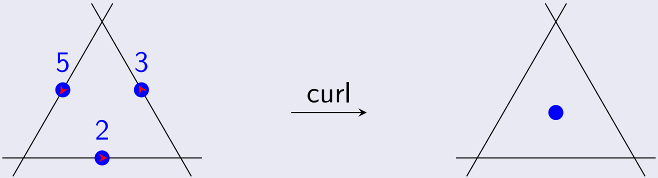
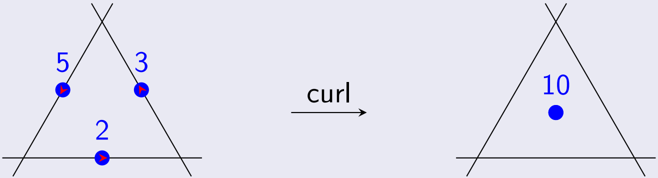
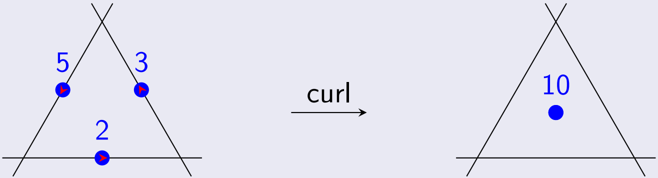


Discrete curl



Discrete curl



Green's/Stokes's Theorem

$$\int_S \operatorname{curl} E = \int_C E$$

where C is the boundary of the surface S.

From Euler characteristic to cohomology (1930s)

The continuous complex (de Rham complex)

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

From Euler characteristic to cohomology (1930s)

The continuous complex (de Rham complex)

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

The discrete complex (simplicial cochain complex)

$$\begin{matrix}\text{discrete}\\\text{scalar fields}\end{matrix} \xrightarrow{\text{grad}} \begin{matrix}\text{discrete}\\\text{vector fields}\end{matrix} \xrightarrow{\text{curl}} \begin{matrix}\text{discrete}\\\text{scalar fields}\end{matrix}$$

From Euler characteristic to cohomology (1930s)

The continuous complex (de Rham complex)

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

The discrete complex (simplicial cochain complex)

$$\begin{array}{ccccc} \text{discrete} & & \text{discrete} & & \text{discrete} \\ \text{scalar fields} & \xrightarrow{\text{grad}} & \text{vector fields} & \xrightarrow{\text{curl}} & \text{scalar fields} \\ \mathbb{R}^V & \longrightarrow & \mathbb{R}^E & \longrightarrow & \mathbb{R}^F \end{array}$$

From Euler characteristic to cohomology (1930s)

The continuous complex (de Rham complex)

$$\text{scalar fields} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{scalar fields}$$

The discrete complex (simplicial cochain complex)

$$\begin{array}{ccccc} \text{discrete} & & \text{discrete} & & \text{discrete} \\ \text{scalar fields} & \xrightarrow{\text{grad}} & \text{vector fields} & \xrightarrow{\text{curl}} & \text{scalar fields} \\ \mathbb{R}^V & \longrightarrow & \mathbb{R}^E & \longrightarrow & \mathbb{R}^F \end{array}$$

Theorem (De Rham's Theorem, 1931)

de Rham cohomology equals simplicial cohomology

From Euler characteristic to cohomology (1930s)

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Theorem (De Rham's Theorem, 1931)

de Rham cohomology equals simplicial cohomology

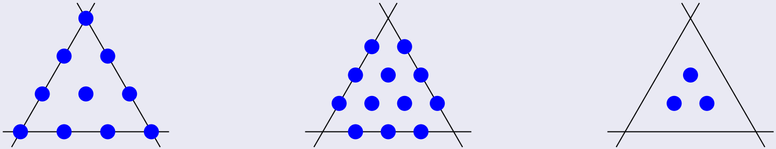
Corollary (Euler characteristic)

$$V - E + F = \dim H^0 - \dim H^1 + \dim H^2$$

Back to finite elements

We've already seen a different discrete complex

continuous piecewise cubic scalar fields $\xrightarrow{\text{grad}}$ tangentially continuous piecewise quadratic vector fields $\xrightarrow{\text{curl}}$ discontinuous piecewise linear scalar fields



$$\mathbb{R}^{V+2E+F} \longrightarrow \mathbb{R}^{3E+3F} \longrightarrow \mathbb{R}^{3F}$$

Back to finite elements

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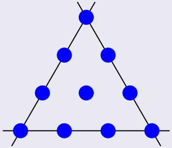
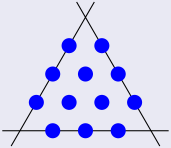
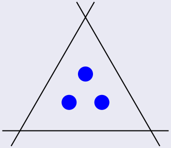






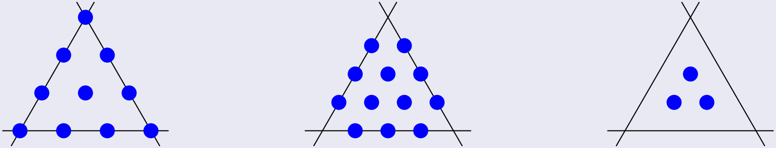
$$\mathbb{R}^{V+2E+F} \longrightarrow \mathbb{R}^{3E+3F} \longrightarrow \mathbb{R}^{3F}$$

Euler characteristic and cohomology

Back to finite elements

We've already seen a different discrete complex

continuous piecewise cubic scalar fields $\xrightarrow{\text{grad}}$ tangentially continuous piecewise quadratic vector fields $\xrightarrow{\text{curl}}$ discontinuous piecewise linear scalar fields



$$\mathbb{R}^{V+2E+F} \longrightarrow \mathbb{R}^{3E+3F} \longrightarrow \mathbb{R}^{3F}$$

Euler characteristic and cohomology

- We saw this complex has the right Euler characteristic:

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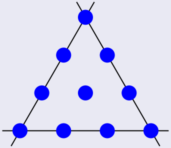
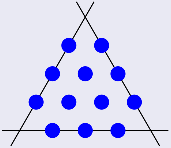
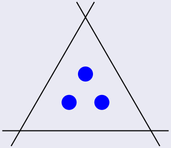
Back to finite elements

We've already seen a different discrete complex

continuous piecewise cubic scalar fields $\xrightarrow{\text{grad}}$ tangentially continuous piecewise quadratic vector fields $\xrightarrow{\text{curl}}$ discontinuous piecewise linear scalar fields

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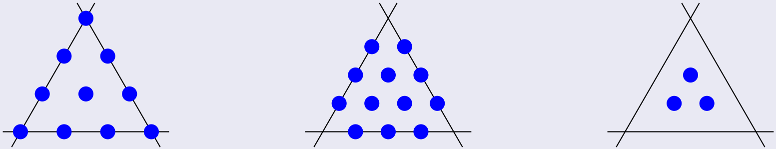

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- Moreover, the cohomology is right, too.
  - That's why the spaces work well (Arnold, Falk, Winther, 2006).

Can we interpret simplicial cochains as finite elements?

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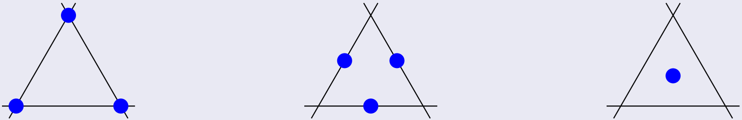

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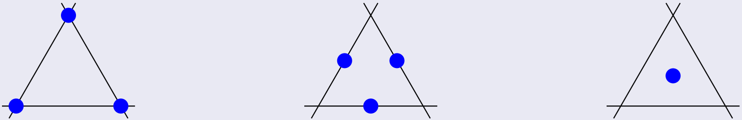

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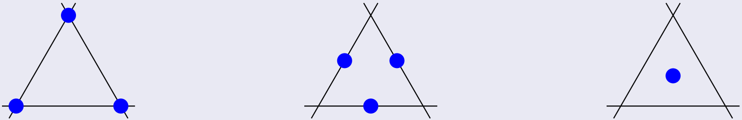

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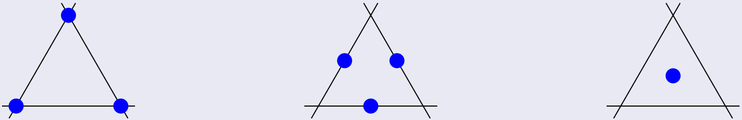

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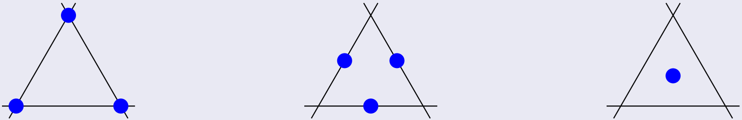

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A modern language for vector calculus

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- Vector calculus:

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$\mathcal{P}_r\Lambda^n(\mathcal{T})$	discontinuous piecewise polynomial scalar fields

Complexes revisited

We've seen

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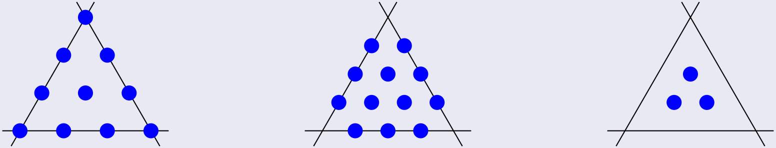
$$\begin{array}{c}\text{continuous}\\ \text{piecewise cubic}\\ \text{scalar fields}\end{array} \xrightarrow{\text{grad}} \begin{array}{c}\text{tangentially continuous}\\ \text{piecewise quadratic}\\ \text{vector fields}\end{array} \xrightarrow{\text{curl}} \begin{array}{c}\text{discontinuous}\\ \text{piecewise linear}\\ \text{scalar fields}\end{array}$$



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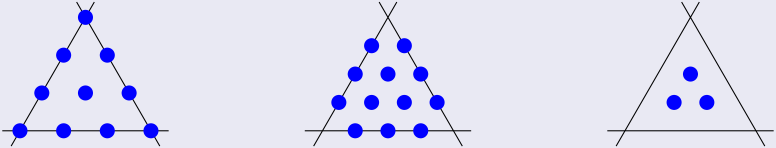

$$\mathcal{P}_3\Lambda^0(\mathcal{T}) \xrightarrow{\quad d \quad} \mathcal{P}_2\Lambda^1(\mathcal{T}) \xrightarrow{\quad d \quad} \mathcal{P}_1\Lambda^2(\mathcal{T})$$

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Duality between $\mathcal{P}$ and $\mathcal{P}^-$

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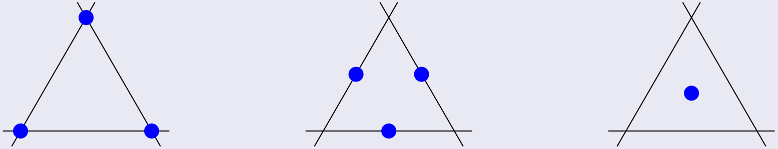

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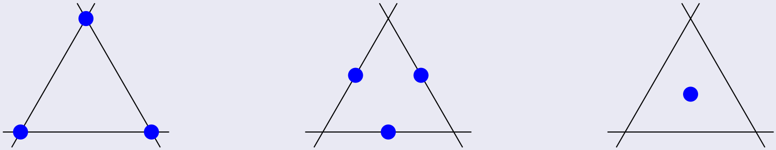

$$\mathcal{P}_1^-\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_1^-\Lambda^1(\mathcal{T}) \xrightarrow{d} \mathcal{P}_1^-\Lambda^2(\mathcal{T})$$

More complexes

Theorem (Arnold, Falk, Winther, 2006)

For a triangulation $\mathcal{T}$, the cohomology of the complexes

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Theorem (Arnold, Falk, Winther, 2006)

We can "mix and match" using any of the maps

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}), \qquad \mathcal{P}_r\Lambda^k(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T})$$

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Douglas N. Arnold, Richard S. Falk, and Ragnar Winther.
Finite element exterior calculus, homological techniques, and applications.
Acta Numer., 15:1–155, 2006.

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Finite element exterior calculus: from Hodge theory to numerical stability.
Bull. Amer. Math. Soc. (N.S.), 47(2):281–354, 2010.

How do finite element spaces yield numerical methods?

Recall our sample problem

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- Given f, solve (1) for u, where u and v are restricted to be in the finite element space.
- Get a finite-dimensional linear system of equations.

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Eigenvalues of the curl curl operator

On a square domain, find a vector field u (with appropriate boundary conditions) such that $\operatorname{curl}\operatorname{curl} u = \lambda u$.

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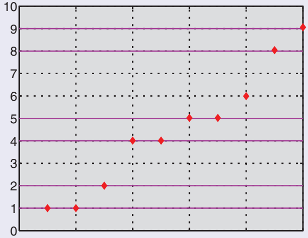

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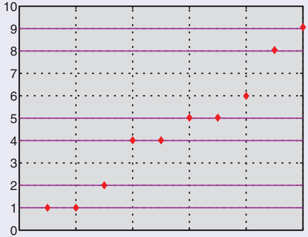

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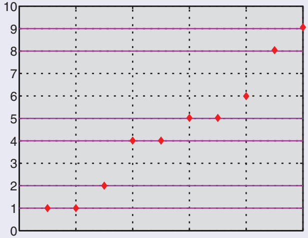

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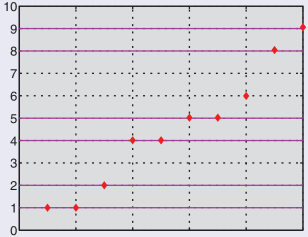

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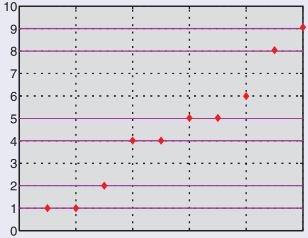

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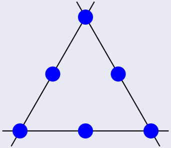
Bases for scalar fields



Bases for scalar fields

- Recall barycentric coordinates:

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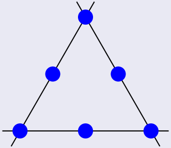

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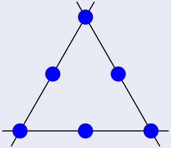

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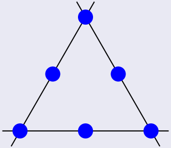

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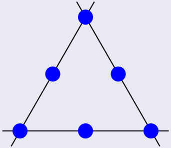

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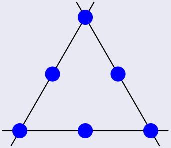

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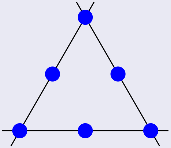

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Further directions

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Thank you