Uncovering the Lagrangian of a system from discrete observations

Yakov Berchenko-Kogan

Massachusetts Institute of Technology

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The Discrete Euler-Lagrange Equations

Given a system with Lagrangian $L(x, v)$, we can discretize the action with time step τ by summing the discrete Lagrangian

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The principle of stationary action yields the discrete Euler-Lagrange equations

$$D_2 L_d(x, y) + D_1 L_d(y, z) = 0,$$

relating any three consecutive points x, y, and z on a discrete trajectory.

Recovering the Discrete Lagrangian

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- For example, if $L(x, y)$ is a discrete Lagrangian, then the Lagrangian

$$L'(x, y) = \alpha L(x, y) + \beta(y^2 - x^2) + \gamma(y - x) + \delta$$

  produces the same discrete Euler-Lagrange equations, for any choice of α, β, γ, and δ.

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- Trajectory data can't distinguish between equivalent Lagrangians, nor would it be useful to do so.

A Second-Order Approximation

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$$L_d \approx a(x-p)^2 + 2b(x-p)(y-p) + c(y-p)^2 + d_p(x-p) + e_p(y-p) + f_p,$$

where $p = (x_0 + y_0)/2$.

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Discrete Euler-Lagrange Equations

For three consecutive points x, y, and z on a trajectory, we can apply the discrete Euler-Lagrange equations $D_2 L_d(x, y) + D_1 L_d(y, z) = 0$ to find

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We will use nearby triplets (x, y, z) from our trajectory measurements to estimate $a+c$, b, and $d_p + e_p$.

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Scaling the Parameters

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$$A := (a + c) \|y_0 - x_0\|, \quad B := 2b \|y_0 - x_0\|, \quad D := d_p + e_p.$$

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Estimating Lagrangian Parameters with Several Data Points

Given data of consecutive triplets (x_i, y_i, z_i), we estimate A, B, and D up to scaling at a point (x_0, y_0) as follows.

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Given data of consecutive triplets (x_i, y_i, z_i), we estimate A, B, and D up to scaling at a point (x_0, y_0) as follows.

- Construct a matrix M whose rows are

$$w_i \cdot \begin{pmatrix} 2y_i - (x_0 + y_0) & x_i + z_i - (x_0 + y_0) & \|y_0 - x_0\| \end{pmatrix}.$$

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- For the correct values of the parameters, we will have $M \begin{pmatrix} A \\ B \\ D \end{pmatrix} \approx 0$.
- Estimate A, B, and D by the eigenvector corresponding to the least eigenvalue of $M^T M$.

Assigning Weights to the Data Points

The matrix of coefficients

The ith row of M is

$$w_i \cdot \left(2y_i - (x_0 + y_0) \quad x_i + z_i - (x_0 + y_0) \quad \|y_0 - x_0\|\right).$$

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Distance

We define the distance bewteen (x_0, y_0) and (x, y) to be

$$\delta((x_0, y_0), (x, y))^2 = \left\| \frac{x+y}{2} - \frac{x_0+y_0}{2} \right\|^2 + \tau_s^2 \left\| \frac{y-x}{\tau} - \frac{y_0-x_0}{\tau} \right\|^2,$$

where τ_s is a parameter and τ is the timestep.

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Weights

$$w_i = \exp\left(-\frac{1}{2\sigma^2} \left(\delta((x_0, y_0), (x_i, y_i))^2 + \delta((x_0, y_0), (y_i, z_i))^2 \right) \right),$$

where σ is another parameter.

The Simple Pendulum

The Lagrangian

$$L_d(x,y) = \tau\left(\frac{1}{2}\left(\frac{y-x}{\tau}\right)^2 - \left(1-\cos\left(\frac{x+y}{2}\right)\right)\right).$$

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True Values of Lagrangian Parameters

Using a Taylor approximation to the Lagrangian, we find that

$$\frac{B}{A} = -\frac{4 + \tau^2 \cos\left(\frac{x_0 + y_0}{2}\right)}{4 - \tau^2 \cos\left(\frac{x_0 + y_0}{2}\right)}, \qquad \frac{D}{A} = -\frac{4\tau^2}{\|y_0 - x_0\|} \cdot \frac{\sin\left(\frac{x_0 + y_0}{2}\right)}{4 - \tau^2 \cos\left(\frac{x_0 + y_0}{2}\right)}.$$

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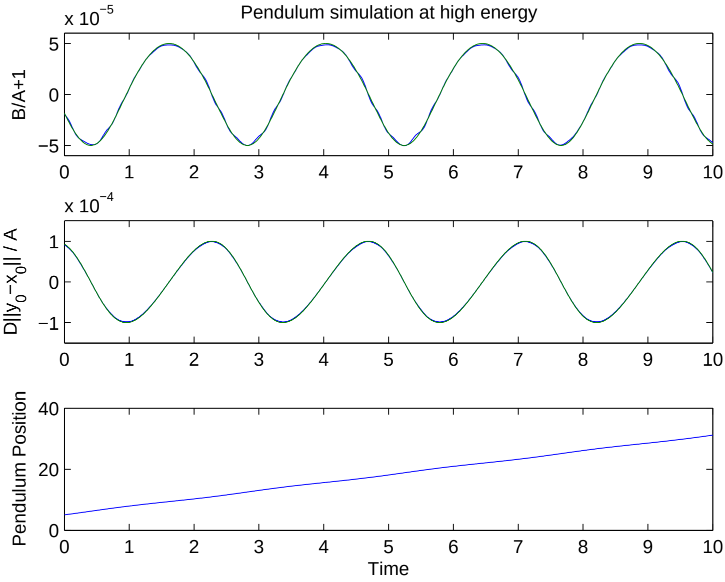
Parameters Computed From Trajectories

I computed the parameters from the trajectories with Matlab. The graphs of $\frac{B}{A}+1$ and $\frac{D}{A}\|y_0-x_0\|$ are on the following slides.



Pendulum simulation at high energy
x 10^-5
B/A+1
5
0
−5
0 1 2 3 4 5 6 7 8 9 10
x 10^-4
D||y0−x0|| / A
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0
−1
0 1 2 3 4 5 6 7 8 9 10
Pendulum Position
40
20
0
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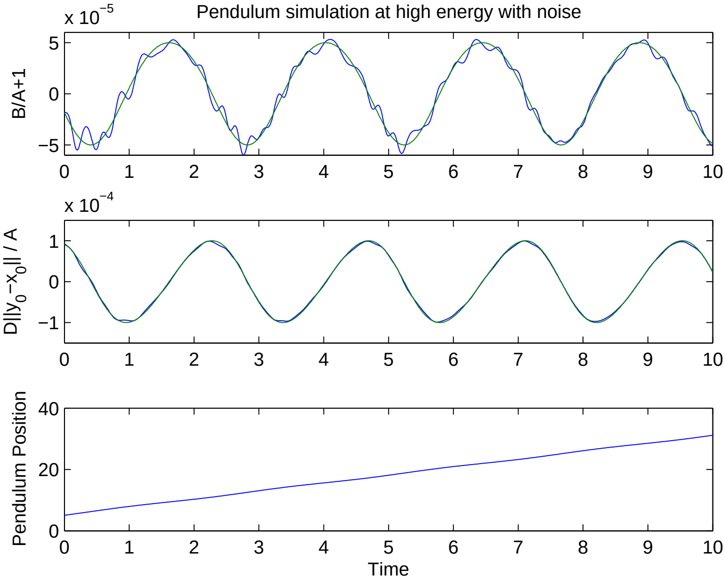
Time



Pendulum simulation at high energy with noise
x 10^−5
B/A+1
5
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−5
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x 10^−4
D||y0−x0|| / A
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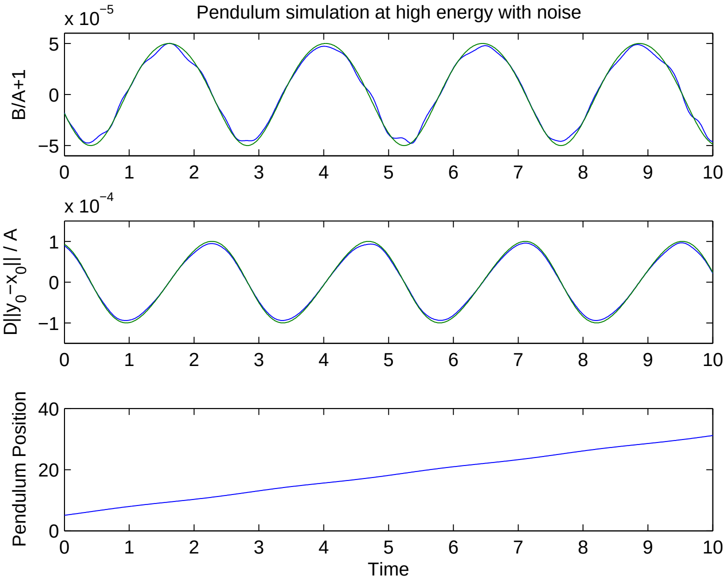
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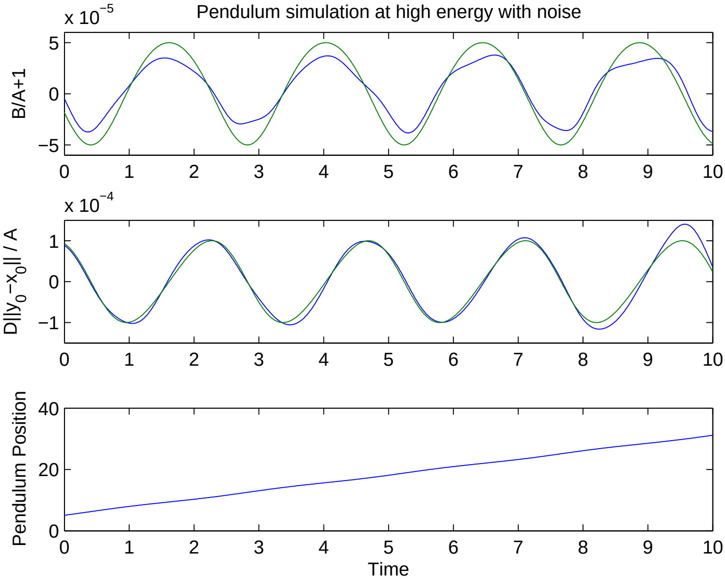
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